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ABSTRACT

These materials on irrational numbers are designed for use in workshops for pre- and in-service teacher training, especially for teachers in junior high schools. The general aims are: to improve teachers' mathematical knowledge of topics, to enrich teachers' mathematical background, to allow opportunity to discuss relevant didactics, and to create a reasonable image of mathematics and mathematical activity as a human endeavor. The worksheets (in booklet form) generally have a brief biographical-chronological introduction to set the historical scene, an historical source, and leading questions on the source materials and on mathematical and didactical consequences thereof. Each worksheet is accompanied by answer sheets (also in booklet form), containing solutions and further sources and information. Six worksheets are contained in the set: the Pythagoreans, Euclid and the Elements, irrationals in the 16th and 17th centuries, Rafael Bombelli, Nicholas Saunderson, and Dedekind and the definition of the irrationals. Comments on how to use the worksheets are briefly presented. (MNS)

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THE WEIZMANN INSTITUTE OF SCIENCE
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The Irrational Numbers

A source-work collection
for in-service and pre-service
teacher courses

A. Arcavi
M. Bruckheimer

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The Irrational Numbers

Introduction and commentaries to the set of worksheets on some episodes in the historical development of the irrational numbers

The attached materials are designed for use in workshops for pre and in-service teacher training, in particular for teachers in junior high schools. Although the sheets have a single topic, viz. the irrational numbers, many general points concerning mathematical activity and its teaching are included (see the individual commentaries).

Thus the general aims of this and other sequences of worksheets is:

1- to *improve* the teachers' *mathematical knowledge* of topics included in the curriculum, and that in a way which motivates the teacher to reconsider topics previously studied but possibly imperfectly understood;

2- to *enrich* the teachers' *mathematical background* to topics in the curriculum;

3- to allow opportunity for the *discussion of relevant didactics* and to consider them in relation to the pure mathematics of the topics concerned;

4- to create a *reasonable image of mathematics* and mathematical activity as a human endeavour. In particular, to create an awareness of the history of topics included in the curriculum.

In general, the worksheets in this sequence have the following form:

- a brief biographical-chronological introduction in order to set the historical scene,
- an historical source; as far as possible a primary source,
- leading questions on the source material and on mathematical and didactical consequences thereof.

To each worksheet an extensive discussion of the solutions, and points arising from them, is given in the respective *answer sheet*. Thus, the answer sheets contain not only the detailed solutions to the questions, but further source material, background, historical and mathematical information. Both the worksheets and the answer sheets are designed as learning materials.

At the present time, this sequence contains 6 worksheets which are intended to be worked in the following order:

1- The Pythagoreans.
2- Euclid and the *Elements*.
3- Irrationals in the 16th and 17th centuries.
4- Rafael Bombelli.
5- Nicholas Saunderson.
6- Dedekind and the definition of the irrationals.

We have used the sheets in the following:

i- In-service workshops.
ii- Pre-service courses.
iii- Correspondence course.
iv- Single worksheets in other workshops.

The work is usually guided by a tutor. Participants are given the first worksheet. They work in groups or individually, with the tutor's "interference" if necessary. Then a collective guided discussion takes place (except for iii-), and the answer sheet is distributed. And so on.

For use with Israeli teachers, we translated the sources in the worksheets freely into Hebrew and gave both the original with the translation, in order to encourage the student to read both (if he understood the original language, if not he could look at it and get some flavour of the period by the form of the print, its elaboration, etc., or even try to identify key words etc. from the translation.) Also the original should be available in cases of misunderstanding attributable to mistranslation. In the present English version there are some extracts in languages other than English, and we include either an "authorized" English translation (for example, Bombelli) or only the English version (for example, Dedekind). Also there are some extracts for which we did not find an "authorized" English translation, in which case we have brought the original only. For English speaking users of these materials, it may be advisable to translate these latter texts freely into English, alongside the original.

This series is not a text. Therefore there is a need for the tutor using the sheets, to add comments, to be in a position to answer questions, and to have in mind the overall structure of the set.

The stages in the history of irrational numbers, represented in these worksheets, are far from being complete. There are certainly, further sheets which could be added, but those here presented give, in our view, the major highlights in the general development of the irrationals. In the following we shall comment briefly on the content of each worksheet, how to link them and some of the points we had in mind when writing and using the worksheets.

Commentary to the individual worksheets

1- The Pythagoreans

This worksheet presents a little from the world of the Pythagoreans: figurate numbers, the pentagram and commensurable line-segments. Then the crisis in the Pythagorean doctrine caused by their realisation of the existence of incommensurable line-segments, is discussed (with two different approaches). Only secondary sources are used in this sheet. This sheet can be enriched by "ramifications"; for example, discussion of several ways to prove the irrationality of $\sqrt{2}$, more details on the golden section, etc.

2- Euclid and *The Elements*

A further stage in the history of the irrationals is the "legitimization of incommensurability". In this context, Euclid's famous masterpiece is presented briefly, and an interesting definition of irrationality is discussed.

3- Irrationals in the 16th and 17th centuries

The (almost) 2000 year jump, can be justified by the sparsity of historical sources that can offer something new, relevant to the development of the irrationals. Nevertheless, it should be worthwhile to refer somehow to this long period. In our workshops, for example, we brought an Hebrew source by Maimonides (1135-1204), one of the greatest Rabbis of all times, codifier, philosopher, physician and astronomer. In his commentary on the Mischna (Eruvin), he discusses the nature of the ratio between the circumference of a circle and its diameter. He states:

which (very briefly) means that "its essence is unknown and cannot be conceived... only by approximation". And then he gives $3\frac{1}{7}$ as an approximate value.

This concern about the nature of irrational numbers became more widespread after decimal fractions were developed in the 16th century, by Simon Stevin (1548-1620), and others, and this is the topic of the present worksheet. Arguments in favour and against irrationals are brought, and it is interesting to note that some of those problems, still remain to trouble students today.

4-5 Rafael Bombelli - N. Saunderson

Once the conflict about irrationals had abated (but not disappeared, in the absence of a mathematical definition), we can consider the next stage as the stage of rational approximations to irrationals. (It should be noted that there were mathematicians who had treated this topic before this period also.)

Two methods of approximation are discussed and compared: Bombelli's and Saunderson's. The first is not so well known by teachers, the second is the most well known (from the algorithmical point of view, but not its justification).

6- Dedekind and the definition of irrationals

The following are examples of stages (between the fifth and sixth worksheets) not treated in this series:

- The rationality or irrationality of some given numbers (for example, the interesting case of π - Lambert's proof of its irrationality in 1766.)
- The existence of algebraic and transcendental irrationals (and also special cases, such as Lindemann's proof of the transcendence of π in 1882, or the existence of number whose transcendence is not yet proved.)

We suggest that these points should at least be mentioned verbally in connection with the Dedekind worksheet. In this worksheet, we bring the original development of the definition, which seems to be easier than its elaboration appearing in many later texts.

Some authors* note the similarity between the Eudoxian theory of proportion (in Euclid's *Elements*, Book V) and Dedekind's definition of a real number as a cut. This may be interpreted as a suggestion that Dedekind got his inspiration from Eudoxus theory, either consciously or not.

In this worksheet, the four elementary operations on irrationals are also defined, and statements like $\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$ are proved.

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* See for example:
Flegg G., *The real numbers*, (History of Mathematics AM 289, Unit 2) The Open University Press, 1974, p.23.

Final Comment

We do not see these sheets as final or definitive. As we find new and or "better" sources, or in the light of experience in the use of the sheets, we make alterations, corrections, replacements, additions, etc.

Other sequences are also in process of development, both with the intention of improving the development per se, and evaluating their implementation.

We would welcome comments from anyone reading these worksheets.

A. Arcavi
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October 1984.

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The Irrational Numbers

Worksheet: The Pythagoreans

One of the most famous names in the history of ancient Greek mathematics is that of Pythagoras, who lived in the sixth century B.C.E. The Pythagorean brotherhood was founded in southern Italy, and engaged in agriculture and scientific and religious philosophy.

It is difficult to asses accurately the contribution they made to mathematics, since there is hardly any contemporary written evidence, and many of the things attributed to the Pythagoreans, are shrouded in legend and hearsay.

Apparently, the basis of their philosophy was *number*. Matter was susceptible to change and therefore not fundamental - fundamental was the numerical aspects in which matter was revealed. They believed that the ordered universe existed because of its harmony, which was expressed by simple numerical relations.

Content of this worksheet

First we learn a little about the world of the Pythagoreans: figurate numbers, properties of the pentagon, commensurable line-segments. Then we discuss the crisis in the

Pythagorean doctrine, when they came up against a discovery which contradicted their world view.

As mentioned above, we do not have source material from that time, therefore we shall use secondary sources; that is, material from the histories of mathematics. In the process, we shall bring two different theories of how the Pythagoreans came upon the discovery which rocked their world.

From the Pythagorean world

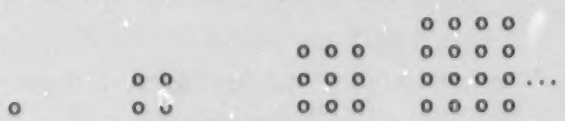
Figurate numbers

In ancient Greece, the decimal number system, with which we are familiar, did not exist. There is no doubt that this is one of the reasons for the fact that the development of arithmetic and algebra lagged far behind that of geometry. A number was denoted by a letter of the alphabet. In addition, apparently, the Pythagoreans represented numbers by dots. This form of representation is the source of the concept of *figurate numbers*.

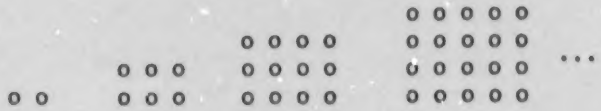
The following are some examples of figurate numbers.

- Triangular numbers are represented as follows.

- Square numbers are formed by dots arranged in a square.



- Rectangular numbers take the form of a rectangle whose length exceeds its breadth by one dot.

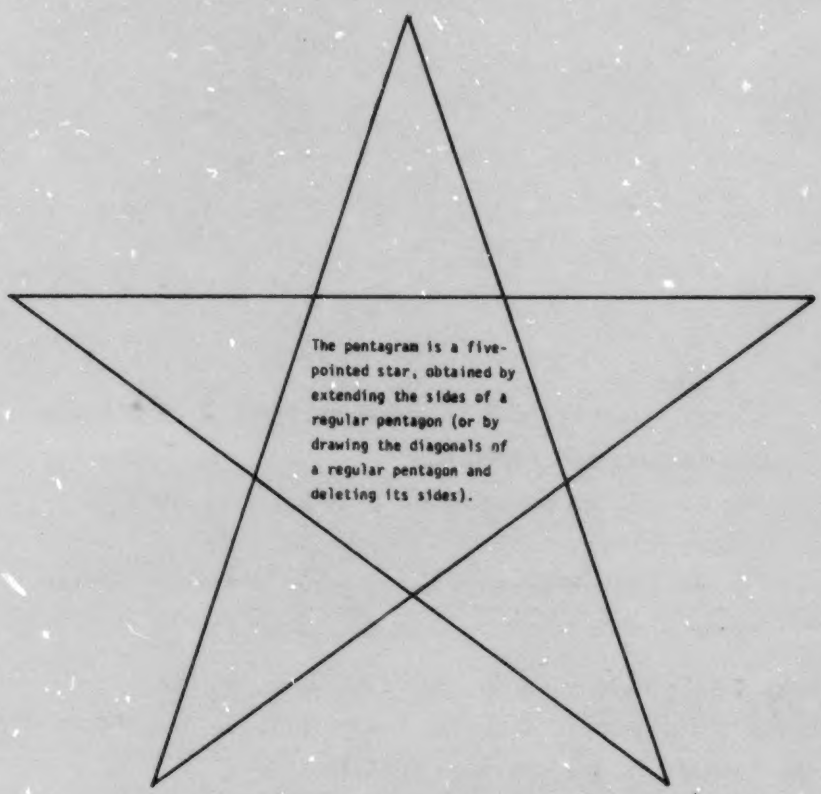


Questions

1. Show by drawing
   a) A square number (n^2) is equal to the sum of all the odd numbers which precede n.
   b) The sum of two consecutive triangular numbers is a square number.
   c) The relation between a rectangular and a triangular number.
2. What the Pythagoreans showed using dots, we can prove using the algebraic techniques available to us. Prove the results in the previous question.

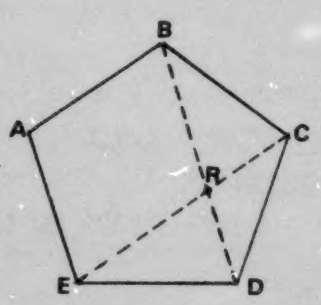
The pentagon

The Pythagoreans, by means of numerical ratios, found particular expression for their concept of harmony in geometry. Apparently, they knew the properties of various regular polygons, including the pentagon and the pentagram.


The pentagram is a five-pointed star, obtained by extending the sides of a regular pentagon (or by drawing the diagonals of a regular pentagon and deleting its sides).

The Pentagram - Badge of the Pythagoreans

3. ABCDE is a regular pentagon.



   a) Show that ΔRCD is isosceles.

   b) Show that the angle CRD is equal to the angle A (or any other angle of the pentagon).

   c) Show that BC = BR.

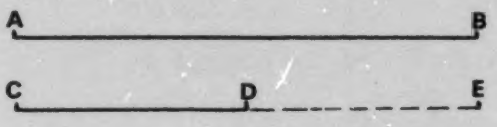
We shall have cause to refer to these results again later.

Common Measure

The Pythagoreans knew how to find the greatest common measure, by the method of mutual subtraction.

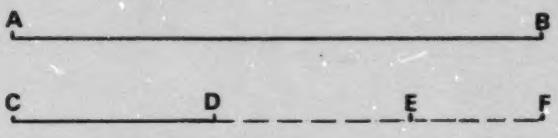
For example,

a)

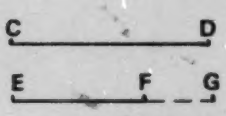


If CD = DE, AB = 2CD. In this case CD is the greatest common measure, and the ratio of CD to AB is the same as the ratio of 1 to 2.

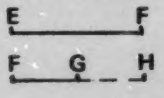
b)



In this case CD is not the common measure. So the process continues



Again EF is not a common measure, so we continue



This time we have found the greatest common measure, and we denote FG by d. Hence

$$CD = 3d$$

$$AB = 8d$$

and the relation between CD and AB is the same as the relation of 3 to 8.

Note: We have exemplified the process of finding the greatest common measure by drawing. Drawing is, of course, not accurate and will have to be replaced by geometrical reasoning.

Question

4. If we draw the two segments as sides of a rectangle, there is another interesting way of illustrating the process of finding the greatest common measure*. (The dashed curves are drawn with a compass.)

Denote the greatest common measure by a, and mark it appropriately on the figure. Then fill in the following blanks.

AB = ____ a , AC = ____ a

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* See for example,
Artmann B. & Seeger V., Geschichte, Geometrie und Irrationalzahlen. Drei Stunden in der Klasse 9. *Der Mathematikunterricht*, 1982, 28, (4), p. 20-9.

The discovery of incommensurable segments

Can we always find a common measure for any two segments? Apparently the Pythagoreans at one time thought that one could.

The following extract* describes the crisis caused by the discovery of incommensurable segments. It contradicted their view of harmony in the universe, and thus began the history of the irrational numbers.

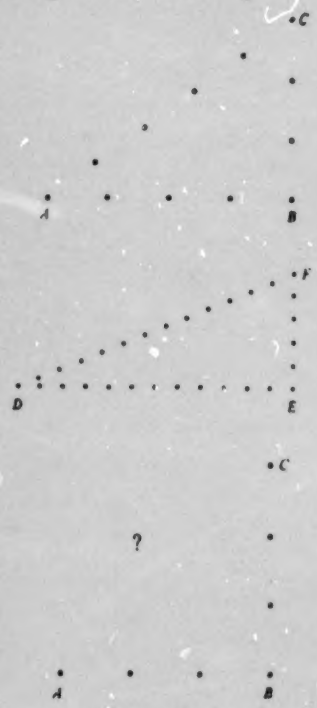
We have seen that the Pythagoreans believed that everything, whether in geometry or in human affairs, could be explained in terms of number. By this they meant whole numbers represented by dots or finite indivisible elements of some sort. Since $3^2 + 4^2 = 5^2$; $5^2 + 12^2 = 13^2$; etc., we can (from the Pythagorean theorem) represent the sides and diagonals of certain rectangles by means of equally spaced dots. We can also in such cases express the *ratio* between diagonal and side as the ratio of a pair of whole numbers. Thus, $AC/AB = 5/4$, $DF/DE = 13/12$, ... In such cases we say that the side and diagonal are *commensurable*.

Exactly at what stage the Pythagoreans became aware of the existence of pairs of magnitudes which could not be compared by means of the ratios of whole numbers in this way we do not know. There is a story that a certain Hippasus of Metapontum (*c.* 400 BC) was expelled from the brotherhood and subsequently drowned at sea as a judgement on him for discovering or disclosing the existence of such quantities. For example, in any square, if AC is the diagonal and AB is the side, can we write $AC/AB = m/n$, where m and n are whole numbers? Aristotle refers to a proof that the diagonal and side of a square are *incommensurable* (i.e. $AC/AB \neq m/n$, where m and n are whole numbers) by means of *odds* and *evens*, and since the properties of odd and even numbers played a central role in Pythagorean thinking it is highly probable this was the proof they found: Let $AC/AB = m/n$, where m and n are whole numbers with no common factor; it follows that m and n cannot *both* be *even* numbers. Since $AB = BC$ and $AC^2 = AB^2 + BC^2 = 2AB^2$, then $AC^2/AB^2 = 2 = m^2/n^2$, $2n^2 = m^2$, and m^2 is *even*; but, if m^2 is *even*, then m is even (since the square of an *odd* number is *odd*). Let $m = 2r$, $m^2 = 4r^2 = 2n^2$, $2r^2 = n^2$, n^2 is *even* and n is also *even*. ...

The proof we have given is one of the easiest example of *proof by contradiction* or *reductio ad absurdum*.[1] Such proofs were constantly used in Greek mathematics and indeed some of the finest proofs in mathematics today take this form. ...

According to Plato, Theodorus of Cyrene (*c.* 390 BC), proved that the sides of squares of areas 3, 5, ..., 17 are incommensurable with the sides of squares of area 1. Explanations as to why Theodorus stopped at 17 vary. ...

It follows that a number system consisting of whole numbers and ratios of whole numbers was seen to be insufficient to represent relations between continuous quantities such as line segments, surfaces and volumes. ... Very briefly, the impact of all this was that formal Greek mathematics turned away from the arithmetic emphasis of the Pythagoreans and became entirely *geometric* ...



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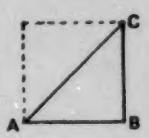
* Taken from

Baron, M.E., *Greek Mathematics*. (AM 289 C1)
The Open University Press, 1974, p. 23-24.

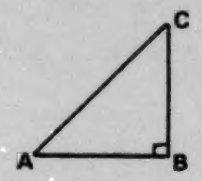
Questions

5. In the following table we bring the "Greek proof" that the diagonal of a square is incommensurable with its side - or, in other words, that $\sqrt{2}$ is irrational. Complete the argument in the table as indicated.

Data	Required to prove	Proof	Reason
The square 	The diagonal and side of the square are segments without _____	Either $\frac{AC}{AB} = \frac{m}{n}$ or $\frac{AC}{AB} \neq \frac{m}{n}$	
$AB = BC$	That is, integers m and n do not exist, such that $\frac{AC}{AB} = \frac{m}{n}$	Suppose $\frac{AC}{AB} = \frac{m}{n}$	
$AB^2 + BC^2 =$ ____		m and n are integers with no common factor.	→ _____
That is $2\,AB^2 =$ _____		$\frac{AC^2}{AB^2} = 2$	
		⇓	

6. In the extract it is stated that Theodorus of Cyrene proved that the sides of squares of areas 3, 5, ..., 17 are incommensurable with the sides of the square of area 1. We can construct the corresponding segments using a compass and an initial triangle.

i) Draw an isosceles right-angled triangle - as in the figure.



We take AB as the unit of measurement; i.e., we denote its length by 1. What is the length of AC?

ii) On AC, construct the perpendicular CC_1 of length 1. Join AC_1.
What is the length of AC_1?

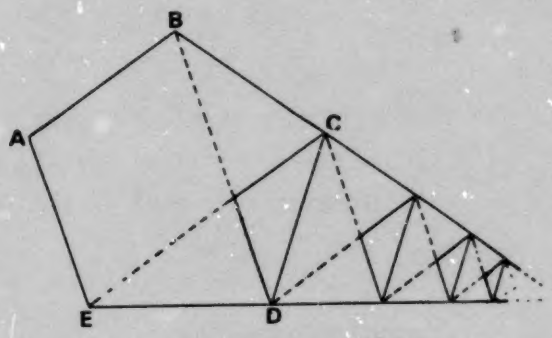
iii) Continue the same construction of unit perpendiculars C_1C_2, C_2C_3, etc. to obtain a sort of spiral, until the drawing begins to overlap. What is the length of each of AC_2, AC_3, ...?

iv) There are historians who maintain that Theodorus knew the proof for a square of area 2 (i.e. that $\frac{m}{n} \neq \sqrt{2}$). Assuming that he used proofs similar to that for 2, reconstruct (in modern symbolism) the proofs for 3, 5 and 8.

There is a theory* that the relation of the side to the diagonal of a square was not the first instance of imcommensurability, which the Greeks met. According to this view, the problem first occurred in the case of a pentagon. In the pentagon, it is geometrically simpler to show that there are two incommensurable segments. (Remember that the above proof for the square looks simple because of our use of modern notation, and the argument is essentially algebraic.)

Questions

7. Using the following figure and questions 3 and 4, show that the side (e.g. BC) and the diagonal (e.g. BD) of a regular pentagon are incommensurable.



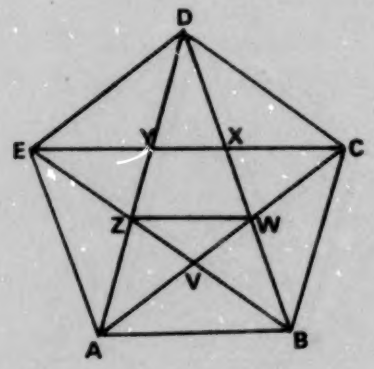
* See, for example

Fritz, K. von, The discovery of incommensurability by Hippasus of Metapontum. *Annals of Mathematics*, 1945, 46 (6), p. 242-64.

8. In the case of the incommensurability of the diagonal and side of a square, $\sqrt{2}$ is involved. The irrationality of which number is implied in the case of the pentagon?

We answer this question in two stages

a)



We take ZV (side of the smaller pentagon) as of unit length. Denote ZW (diagonal of the smaller pentagon) by a. Hence

AZ = (Δ AZW is isosceles)

ZY =

YD =

ED =

AD =

b) Since ABCDE and VWXYZ are regular pentagons, the ratio of the diagonal to the side is the same in both; that is,

$$\frac{AD}{ED} = \frac{ZW}{ZV} .$$

Use part a) to find the answer to our question.

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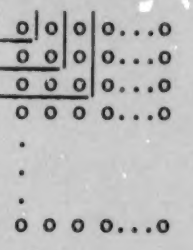
The Irrational Numbers

Answer sheet: The Pythagoreans

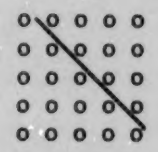
From the Pythagorean world

Figurate numbers

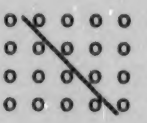
1. a) A square number (n^2) id equal to the sum of all the odd numbers which precede n.



b) The sum of two consecutive triangular numbers is a square number.



c) The sum of two equal triangular numbers is rectangular.



2. a) We can use the formula for the sum of an arithmetic series:

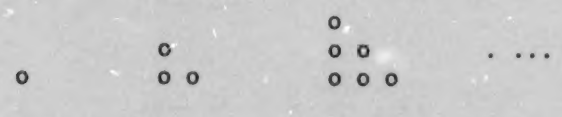
$$S_n = na_1 + \frac{(n - 1)\cdot n\cdot d}{2} ,$$

where, the first term $a_1 = 1$,

the difference between consecutive odd numbers $d = 2$.

Whence $S_n = n + (n - 1)n = n^2$.

b) The representation



shows that the nth triangular number is

$$a_n = 1 + 2 + 3 + \ldots + n .$$

That is $a_n = \frac{n(n + 1)}{2}$

Hence $\frac{n(n + 1)}{2}$ and $\frac{(n + 1)(n + 2)}{2}$ are two consecutive triangular numbers. Their sum is

$$\frac{(n + 1)}{2}(n + n + 2) = \frac{(n + 1)(2n + 2)}{2} = (n + 1)^2$$

c) From the previous section

$$2\,a_n = n(n + 1)$$

which is triangular.

There are more (plane) figurate numbers, pentagonal, hexagonal, etc. For example, the sequence of pentagonal numbers is 1, 5, 12, 22,...

1 5 12

I.e. 1 1 + 4 1+4+7

$$a_n = 1 + 4 + \ldots + (3n - 2) = \frac{3n^2 - n}{2}.$$

The following are general rules for the construction of any (plane) figurate number.

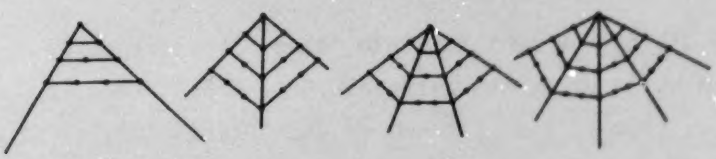
a) The number 1 is figurate for any type of figure.

b) Construct the second figurate number of the chosen type.

c) Choose one vertex of the polygon in b) and extend (in the direction of the adjacent vertices) the two sides meeting at that vertex by one point.

d) Construct the regular polygon of the chosen type on the two extended sides obtained in c).

An alternative method for constructing figurate numbers is illustrated in the following diagrams.



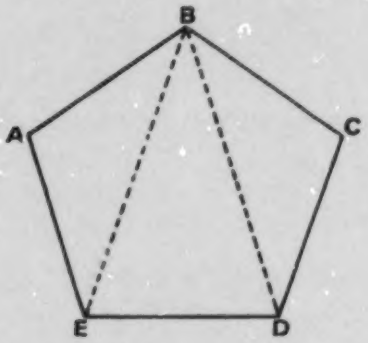
There are also 3-dimensional figurate numbers. For example, cubic numbers 1, 8, 27, ...

For further reading on figurate numbers see, for example,

- Bunt L.N.H., Jones P.S. and Bedient J.D., *The Historical Roots of Elementary Mathematics*, Prentice Hall, 1976, p.75ff.
- Boyer C.B., *A History of Mathematics*, J.Wiley & Sons, 1968, p. 59.

The pentagon

3. a)



There are various ways of proving the required results. We shall use two theorems which were, apparently, known to the Pythagoreans*.

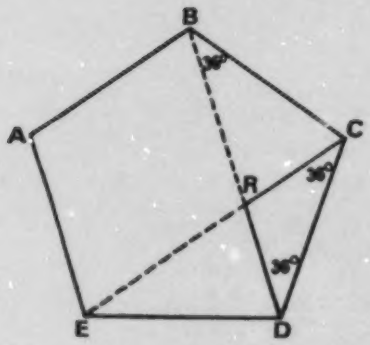
I - In an isosceles triangle the base angles are equal. And, conversely, if there are two equal angles in a triangle, then it is isosceles.

II - The sum of the (internal) angles of a triangle is two right angles (180^0).

If we divide the pentagon into three non-overlapping triangles (e.g. by the diagonals BD and BE), then it follows from II that the sum of the internal angles of the pentagon is 540^0. Since the pentagon is regular, each angle is 108^0. Further Δ BCD is isosceles, so the base angles are equal and each of them is 36^0.

---

* Fritz R. von, The discovery of incommensurability by Hippasus of Metapontum, *Annals of Mathematics*, 1945, 46, (6), p. 242-64.



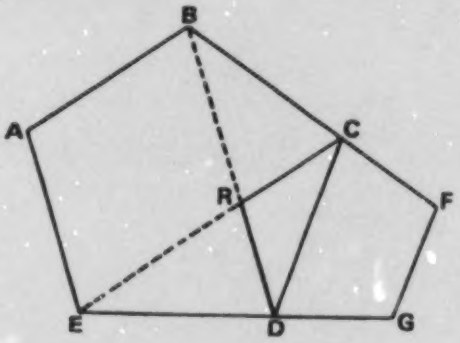
For the same reason (Δ CDE) angle ECD is also 36^0.
Hence Δ RCD is isosceles.

b) From the angle sum in Δ CRD, we obtain

$$DRC = 108^0 = \measuredangle A .$$

c) $BCR = 108^0 - 36^0 = 72^0$, and from the angle sum in Δ BCR, $\measuredangle$ BRC is also equal to 72^0. Hence Δ BCR is isosceles, i.e. $BC = BR$.

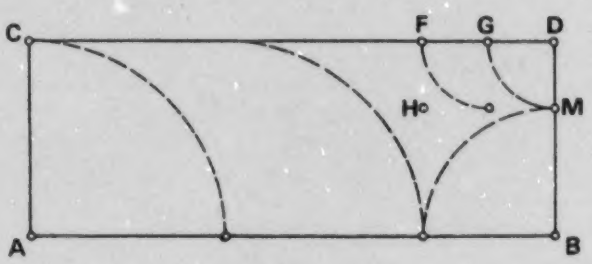
Since $RC = RD$ and $\measuredangle CRD = 108^0$, we can construct a second regular pentagon RCFGD as shown in the following figure.



We shall use this construction in Question 7.

Common measure

4.



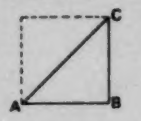
The segment FG (or FH) is the common measure. Since FD is equal to MB, we have

$$AC = 3a \quad \text{and} \quad AB = 8a \ .$$

Hence the ratio of the segment AC to the segment AB is as 3 to 8.

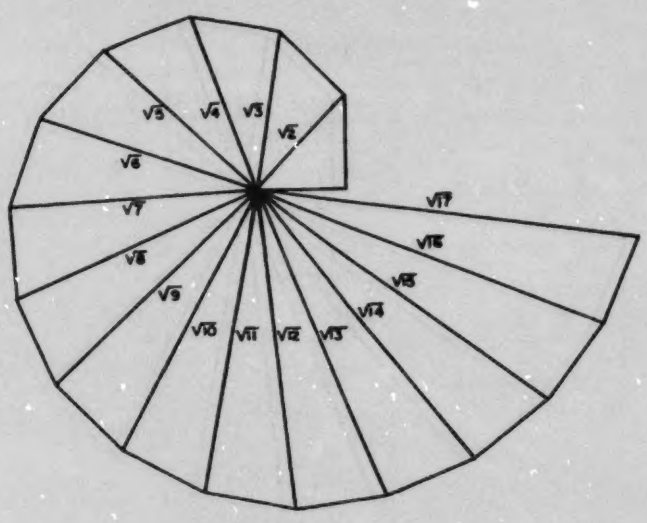
The discovery of incommensurable segments

5.

Data	Required to prove	Proof	Reason
The square:  $AB = BC$ $AB^2 + BC^2 = AC^2$ That is $2AB^2 = AC^2$	The diagonal and side of the square are segments without a common measure. That is integers m and n do not exist, such that $\frac{AC}{AB} = \frac{m}{n}$	Either $\frac{AC}{AB} = \frac{m}{n}$ or $\frac{AC}{AB} \neq \frac{m}{n}$ Suppose $\frac{AC}{AB} = \frac{m}{n}$ m and n are integers with no common factor. ⇓ $\frac{AC^2}{AB^2} = 2$ ⇓ $\frac{m^2}{n^2} = 2$ ⇓ $m^2 = 2n^2$ ⇓ m^2 is even 2 ⇓ m is even ⇓ $m = 2r$ ⇓ $m^2 = 4r^2$ ⇓ $2n^2 = 4r^2$ ⇓ $n^2 = 2r^2$ ⇓ n^2 is even ⇓ n is even ⇓ n and m have a common factor which contradicts our assumption above.	For if they had a common factor, we could always cancel and obtain m and n without a common factor. See data By assumption m is either even or odd. Assume that it is odd. Then $m = 2k + 1$ for some $k \in N$. Hence $m^2 = 4k^2 + 4k + 1$ which is odd, and hence contradicts the previous conclusion that m^2 is even.

6. In his book *Theatetus*, Plato states that Theodorus proved that the sides of squares whose area is 3,5,..., 17 do not have a common measure with the side of the square of unit area. Unfortunately he does not say how. Why did he stop at 17? This question has interested many historians of Greek mathematics.

   i) - iii) There is a theory (brought by Heath*, but he rejects it) that the meaning of "proved" is no more than that he demonstrated by construction, that such segments actually exist. The required construction is shown in the following figure.



We see that the next step, $\sqrt{18}$, overlaps the first. This could be one of the possible reasons for Theodorus' stopping at 17.

---

* Heath T., *A History of Greek Mathematics*, Vol I, Oxford, 1921, p. 203-7.

iv) Hardy* suggests that it is possible that Theodorus, dealt with each case separately. In which case, to prove that the side of a square of area 3 does not have a common measure with the side of a unit square, one has to show that

$$\frac{m^2}{n^2} \neq 3 .$$

The proof is similar to that in Question 5, except that in this case, we have to show that

$$m^2 \text{ a multiple of } 3 \implies m \text{ a multiple of } 3 .$$

As before, assume that m is not a multiple of 3, then either

$$m = 3k - 1 \quad \text{or} \quad m = 3k + 1 , \; k \in N$$

Hence $m^2 = 9k^2 - 6k + 1$ or $m^2 = 9k^2 + 6k + 1$

$= 3(3k^2 - 2k) + 1$ $= 3(3k^2 + 2k) + 1$

In either case m^2 is of the form $3\ell + 1$, $\ell \in N$, and hence cannot be a multiple of 3, which contradicts our previous assumption.
Hence we conclude that m is a multiple of 3, and the remainder of the proof is exactly as in Question 5.

---

* Hardy G.H. & Wright E.M., *An Introduction to the Theory of Numbers*, Oxford, 1945, p. 42-5.

The proof for the square of area 5 is again similar. This time we have to show that

$$m^2 \text{ a multiple of } 5 \implies m \text{ a multiple of } 5$$

Assume that, on the contrary,

$$m^2 \text{ a multiple of } 5 \implies m \underline{\text{not}} \text{ a multiple of } 5,$$

then m must be of the form

$$m = 5k + r, \quad k \in N, \quad r = 1, 2, 3, 4.$$

Hence $$m^2 = 5k^2 + 10kr + r^2$$
$$= 5(5k^2 + 2kr) + r^2$$

Since r^2 is equal to one of 1, 4, 9, 16, none of which are multiples of 5. Hence, once again, the contradiction.

If we attempt a similar proof for the square of area 8, a new point appears. As before, we have to show that

$$m^2 \text{ a multiple of } 8 \implies m \text{ a multiple of } 8$$

Assume that, on the contrary,

$$m^2 \text{ a multiple of } 8 \implies m \underline{\text{not}} \text{ a multiple of } 8$$

then m must be of the form

$$m = 8k + r, \quad k \in N, \quad r = 1, 2, \ldots, 7.$$

Hence $$m^2 = 64k^2 + 16kr + r^2$$
$$= 8(8k^2 + 2kr) + r^2.$$

Only this time r^2 can be equal to any one of 1, 4, 9, 16, 25, 36, 49, and if $r^2 = 16$, we do not get a contradiction with our assumption that m^2 is a multiple of 8. This does not mean that a contradiction does not exist, only that this method does not lead us to one.

In fact, we go right back to the beginning of the argument, we want to show that

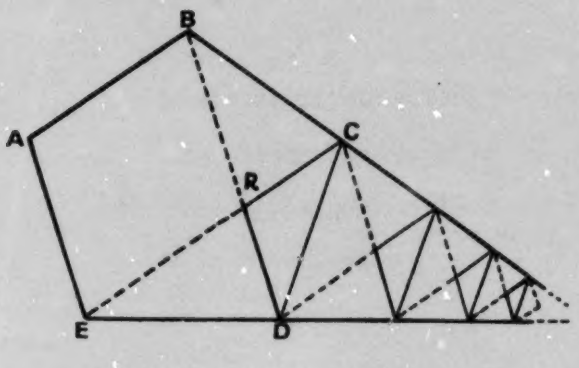
$\frac{m^2}{n^2} \neq 8$. Assume that $\frac{m^2}{n^2} = 8$, i.e.

$$m^2 = 8n^2$$

Then $$m^2 = 2(2n)^2$$

and this is <u>exactly</u> the same situation as in the proof of the square of area 2 .

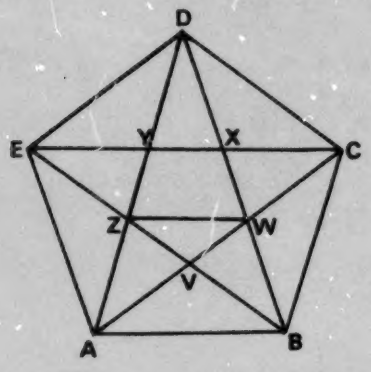
7.



In this case, to show that BC and BD do not have a common measure, we use the method of mutual subtraction, described earlier (pages 4 - 6) in the worksheet. Since BC = BR (Question 3), the problem is equivalent to finding the common measure of BR and BD. Hence, the next step is to find the common measure of BR and RD. But BR = CD, so that the problem is reduced to finding the common measure of RD and CD, which are, once again, the side and diagonal of a regular pentagon. Since this process can be continued indefinitely, we conclude that the

side and diagonal of a regular pentagon do not have a common measure.

8. a)



AZ = a , since Δ AZW is isosceles

ZY = 1 , side of small pentagon

YD = a , Δ EZA is congruent to Δ EYD

ED = 1 + a , ED = DY + YZ (Qu. 3)

AD = 1 + 2a .

b) $$\frac{1 + 2a}{1 + a} = \frac{a}{1}$$

That is $a^2 - a - 1 = 0$

This equation has one positive root $\frac{\sqrt{5} + 1}{2}$.

This number is known as the *golden section* and denoted by the Greek letter ϕ (phi).
It has many interesting properties, some of which are included in a book on the subject, *De Divina Proportione*, by the Italian Luca Pacioli, published in 1509.

The following are a few of its properties:

- The golden section is the only positive number such that the difference between it and its multiplicative inverse is 1.
- The following are two infinite representations of the golden section.

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}} \qquad \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$$

- The Fibonacci sequence (named after the Italian thirteenth century mathematician, Leonardo of Pisa, who was also known by the pseudonym, Fibonacci), is defined as follows:

$$a_1 = 1$$
$$a_2 = 1$$
$$a_n = a_{n-1} + a_{n-2}$$

That is, each term in the sequence is obtained as the sum of the two preceding terms, the first two terms being 1. Thus, the first few terms of the sequence are

1, 1, 2, 3, 5, 8, 13, 21, ...

The ratio $\frac{a_n}{a_{n-1}}$, as n tends to infinity, tends to the golden section.

- The designation "section" arose because of the occurrence of the number in a geometrical context:-

If a line segment AB be divided at a point D, such that the ratio of the whole (AB) to the larger of the sub-segments (AD, say), is equal to the ratio of AD to DB, then this ratio is again $\emptyset$.

Notice that, in the figure of the pentagon in Question 8, Y divides AD in this ratio, i.e. $\frac{AD}{AY} = \frac{AY}{YD}$. Similarly Z divides AD in the same ratio. Further Y divides ZD (and Z divides AY) in the same ratio.

Further details on the golden section can be found, for example, in

Gardner M., *The 2nd Scientific American Book of Mathematical Puzzles and Diversions*, Simon & Schuster N.Y., 1961, p. 89-103.

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The Irrational Numbers

Worksheet: Euclid and *The Elements*

Incommensurables

Very little is known about Euclid. He lived about the year 300 B.C.E. and apparently taught in Alexandria. He is famous as the author of the *Elements*, arguably the best known book in the history of mathematics. Since the invention of printing there have been numerous editions in many languages - and many adaptations. A standard English edition is that of Heath*, who added a scholarly commentary. The following extracts are to set the scene. The first extract is taken from Heath's introduction.

---

* Heath T.L., *The Thirteen Books of Euclid's Elements Translated with Introduction and Commentary.* Dover Pub., N.Y., 1956.

As in the case of the other great mathematicians of Greece, so in Euclid's case, we have only the most meagre particulars of the life and personality of the man.

Most of what we have is contained in the passage of Proclus' * summary relating to him, which is as follows :

> ... Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and ** also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors ...

> ... It is most probable that Euclid received his mathematical training in Athens from the pupils of Plato; for most of the geometers who could have taught him were of that school, and it was in Athens that the older writers of elements, and the other mathematicians on whose works Euclid's *Elements* depend, had lived and taught.

From Heath***

> This wonderful book, with all its imperfections, which indeed are slight enough when account is taken of the date at which it appeared, is and will doubtless remain the greatest mathematical text-book of all time. . . .

---

* Proclus (410-485 C.E.) is one of the commentators through whom many details of Greek mathematics have been preserved.

** To Eudoxus and Theaetetus are attributed many ideas which are brought in the *Elements*, in particular those in Book X, from which we quote later.

*** Heath T.L., *Greek Mathematics*, Vol. I, Oxford Univ. Press, 1921, p. 357-358.

From Boyer*

The *Elements* is divided into thirteen books or chapters, of which the first half dozen are on elementary plane geometry, the next three on the theory of numbers, Book X on incommensurables, and the last three chiefly on solid geometry. There is no introduction or preamble to the work, and the first book opens abruptly with a list of twenty-three definitions. The weakness here is that some of the definitions do not define, inasmuch as there is no prior set of undefined elements in terms of which to define the others. Thus to say, as does Euclid, that "a point is that which has no part," or that "a line is breadthless length," or that "a surface is that which has length and breadth only," is scarcely to define these entities, for a definition must be expressed in terms of things that precede, and are better known than the things defined. ...

Following the definitions, Euclid lists five postulates and five common notions. Aristotle had made a sharp distinction between axioms (or common notions) and postulates; the former, he said, must be convincing in themselves -- truths common to all studies - but the latter are less obvious and do not presuppose the assent of the learner, for they pertain only to the subject at hand ...

... In its time the *Elements* evidently was the most tightly reasoned logical development of elementary mathematics that had ever been put together, and two thousand years were to pass before a more careful presentation occurred. During this long interval most mathematicians regarded the treatment as logically satisfying and pedagogically sound.

---

* Boyer C.B., *A History of Mathematics*, J. Wiley & S., 1968, p. 115-118.

Question

1. The extract from Boyer distinguishes between definitions, postulates and common notions (axioms). Decide which of these three is the appropriate description for each of of the following statements.*

   a) There exists only one straight line containing two given points.

   b) The whole is greater than the part.

   c) Things which are equal to the same thing are also equal to one another.

   d) All right angles are equal to one another.

   e) An obtuse angle is an angle greater than a right angle.

---

* The sentences are taken from Heath's English version of the *Elements*.

The following extract (from Heath's edition of the *Elements*), contains examples of definitions taken from Book X, dealing with incommensurables.

DEFINITIONS.

1. Those magnitudes are said to be **commensurable** which are measured by the same measure, and those **incommensurable** which cannot have any common measure.

2. Straight lines are **commensurable in square** when the squares on them are measured by the same area, and **incommensurable in square** when the squares on them cannot possibly have any area as a common measure.

3. With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called **rational**, and those straight lines which are commensurable with it, whether in length and in square or in square only, **rational**, but those which are incommensurable with it **irrational**.

Questions

2. Give an example of line-segments which are commensurable in square only.

3. Which of the following statements are correct and which not? Give examples (or counter examples) for each statement.

   a) Segments which are commensurable in length are commensurable in square.

   b) Segments which are incommensurable in square are incommensurable in length.

   c) Segments which are commensurable in square are commensurable in length.

   d) Segments which are incommensurable in length are incommensurable in square.

4. Given the following four squares

Square	Area
ABCD	1
EFGH	2
KLMN	3
PQRS	4

We choose the diagonal of the square PQRS as the "assigned straight line (rational)". Which of the following segments are rational and which irrational?

a) The diagonal of the square EFGH.

b) The diagonal of the square KLMN.

c) The diagonal of the square ABCD.

d) The side MN.

e) The side PQ.

f) The segment formed by the combination of the side KL and the side PQ.

In each case, determine whether the segment is commensurable in length and/or in square, with the given segment.

5. Draw a Venn diagram to represent the relation between rational and irrational segments with respect to a given segment and their commensurability.

6. What is the difference between the concept "rational" ("irrational") as defined by Euclid, and that we use today.

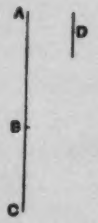
The following is a further extract taken from Heath's edition of Book X.

PROPOSITION 16.

If two incommensurable magnitudes be added together, the whole will also be incommensurable with each of them; and, if the whole be incommensurable with one of them, the original magnitudes will also be incommensurable.

For let the two incommensurable magnitudes AB, BC be added together;

I say that the whole AC is also incommensurable with each of the magnitudes AB, BC.



For, if CA, AB are not incommensurable, some magnitude will measure them.

Let it measure them, if possible, and let it be D.

Since then D measures CA, AB,

therefore it will also measure the remainder BC.

But it measures AB also;

therefore D measures AB, BC.

Therefore AB, BC are commensurable;

but they were also, by hypothesis, incommensurable:

which is impossible.

Therefore no magnitude will measure CA, AB;

therefore CA, AB are incommensurable. [x. Def. 1]

Similarly we can prove that AC, CB are also incommensurable.

Question

7. Proposition 16 contains two statements and the proof of one of them. The proof is given in the following table. Complete it.

Data	To prove	Deduction	In modern algebraic notation
A B C	1) AC and AB are incommensurable	1) There are two possibilities: AB and AC are commensurable or they are not	
Such that AB and BC are incommensurable magnitudes.		Suppose the former...	-------
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The Irrational Numbers

Answer sheet: Euclid and the *Elements*.

1. a) and d) are postulates, since they are related to the subject matter.
   b) and c) are axioms - "truths common to all studies".
   (Note, however, that whereas c) might be described to be universal, b) is not. Thus, in modern times, an infinite set was defined as a set which can be put into one-one correspondence with a subset of itself. For example,

   1, 2, 3, 4, 5, ..., n, ...

   2, 4, 6, 8, 10, ...,2n, ...

   And, in this sense, the "whole is not greater than its part".)
   Finally e) is a definition of a new term.
   (Notice also that d) is a postulate logically linked to the axiom in c).)

   In the first book of the *Elements*, Euclid brings 23 definitions (point, line, circle, etc.), 5 postulates and 5 axioms.
   In the following we bring the 5 postulates as they appear in the Boyer* version.

---

* See
Boyer C.B., *A History of Mathematics*, Wiley, 1968, p.116-7.

Postulates.

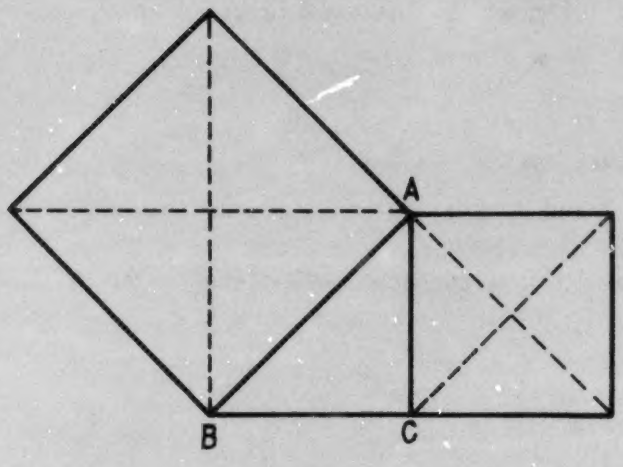
1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. That all right angles are equal.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

The fifth postulate is distinguished from the rest by its complicated enunciation, and it was the most problematic of the five. For some 2000 years, mathematicians tried to prove it and/or to propose alternatives (Heath lists nine), such as

- through a given point, there is just one straight line parallel to a given straight line.
- if a straight line cuts one of two parallel straight lines, then it also cuts the second.

In the end, in the nineteenth century there was an interesting and surprising development. The mathematicians Gauss (1777-1855), Bolyai (1802-1860) and Lobachevskii (1792-1856) showed that it was possible to construct a consistent geometry, and not only one, in which the fifth postulate was negated. Such geometries are called *non-Euclidean*. The existence of such geometries shows that postulate 5 cannot be proved.

2. The simplest example is the diagonal and side of a square. The following figure shows that they are commensurable in square and incommensurable in length (as was shown in the previous worksheet).



The square on AC and the square on AB have a common measure; for example, triangle ABC measures both of them.

3. a) This statement is correct, since if two given lengths have a length which is their common measure, then one can construct a square on it which will measure the squares on the given lengths.

   b) This statement is also correct, since if there is no square which is the common measure of two given squares, there cannot be a length which measures the sides of the two squares.

   In fact, a) and b) are logically equivalent. For if the first is of the form

   $$p \Rightarrow q ,$$

   then the second is of the form

   $$\sim q \Rightarrow \sim p ,$$

   where $\sim p$ denotes the negation of p ("not p"). And for any two statements p, q much that $p \Rightarrow q$, we have $\sim q \Rightarrow \sim p$; i.e.

   $$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$$

c) This is not true, as the example given in the solution of of question 2 shows.

d) This is also not true and again the example in question 2 is sufficient to show this. This is not surprising since c) and d) are logically equivalent.

4. The various lengths and areas are given in the following table.

Square	Area	Side	Diagonal
ABCD	1	1	$\sqrt{2}$
EFGH	2	$\sqrt{2}$	2
KMNL	3	$\sqrt{3}$	$\sqrt{6}$
PQRS	4	2	$\sqrt{8}$

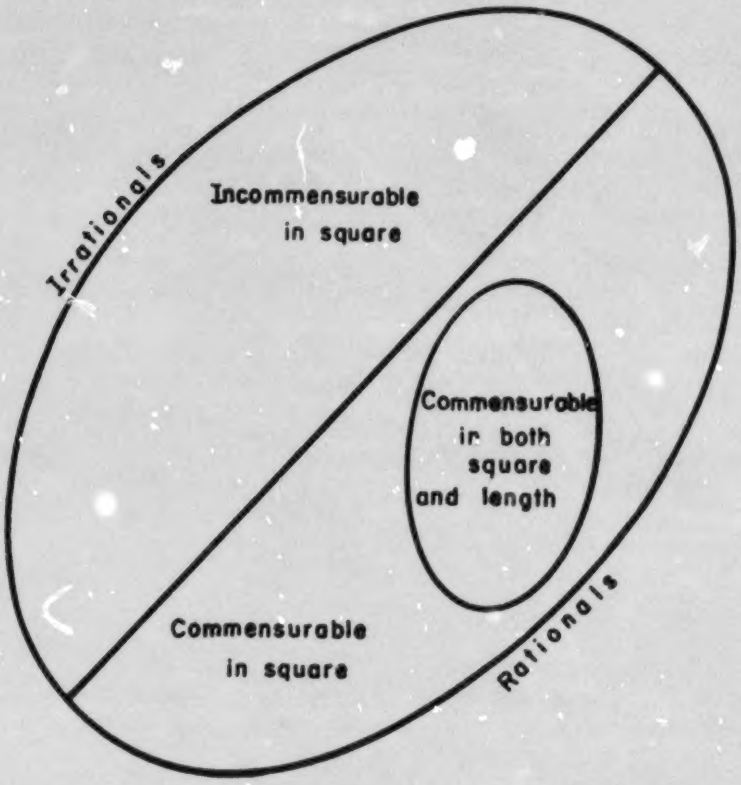
where $\sqrt{8}$ is the "assigned straight line (rational)".

Hence

Length	Rational	Irrational	Commensurable in length	Commensurable in square
a) diagonal EFGH	x	-	-	x
b) diagonal KLMN	x	-	-	x
c) diagonal ABCD	x	-	x	x
d) side MN	x	-	-	x
e) side PQ	x	-	-	x
g) KL + PQ	-	x	-	-

Note that to be rational, a length has only to be commensurable with the given length either in square or in length, and not necessarily both.

5.



Notice that

commensurable in square $\Leftrightarrow$ rational

incommensurable in square $\Leftrightarrow$ irrational ,

whereas this is not true if we replace "square" by "length" (see question 3).

6. Euclid's use of rational and irrational is relative, depending on the choice of "given length", whereas today

it is an absolute classification of numbers. Further, even if we take, for example, the given length to be the unit (1), Euclid's rational includes, among other things, $\sqrt{2}$ because it is commensurable in square with 1.

7.

Data	To prove	Deduction	In modern algebraic notation
Such that AB and BC are incommensurable magnitudes, i.e. there is no measure which measures them both	1) AC and AB are incommensurable magnitudes 2) BC and AC are incommensurable magnitudes	1) There are two possibilities: AB and AC are commensurable or they are not. Suppose the former and that D is the common measure → D will also measure the remainder BC* → AB and BC are commensurable, which contradicts our assumption → AC and AB are incommensurable. 2) Similarly.	Suppose AC=mD, AB=nD, where m and n are natural numbers. BC = AC - AB = (m - n)D

* This was proved in the previous theorem (theorem 15).

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The Irrational Numbers

Worksheet: Irrationals in the 16th and 17th century

In this worksheet we bring extracts* from mathematical texts written in the 16th and 17th century, in which various mathematicians discuss the irrationals.

a) Michael Stifel (1487-1567) was a German mathematician. The mathematical historian Boyer** maintains that his work was of great significance in Germany in the 16th century. The work in question is Stifel's book *Arithmetica Integra* (1544).

The following extract, which is taken from Kline's book***, refers to Stifel's position with regard to irrationals, as expressed in an extract which he brings from the original (translated into English).

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* The extracts are quoted as they appear in other texts. since we had no access to the primary source. Therefore, we note from where the extract is taken and who quotes it.

** Boyer C.B., *A History of Mathematics*, Wiley, 1968, p. 309-310.

*** Kline M., *Mathematical Thought from Ancient to Modern Times*. Oxford U.P., 1972, p. 251.

In his major work, the *Arithmetica Integra* (1544), which deals with arithmetic, the irrationals in the tenth book of Euclid, and algebra, Stifel considers expressing irrationals in the decimal notation. On the one hand, he argues:

> Since, in proving geometrical figures, when rational numbers fail us irrational numbers take their place and prove exactly those things which rational numbers could not prove ... we are moved and compelled to assert that they truly are numbers, compelled that is, by the results which follow from their use—results which we perceive to be real, certain, and constant. On the other hand, other considerations compel us to deny that irrational numbers are numbers at all. To wit, when we seek to subject them to numeration [decimal representation] ... we find that they flee away perpetually, so that not one of them can be apprehended precisely in itself.... Now that cannot be called a true number which is of such a nature that it lacks precision.... Therefore, just as an infinite number is not a number, so an irrational number is not a true number, but lies hidden in a kind of cloud of infinity.

He then argues that real numbers are either whole numbers or fractions; obviously irrationals are neither, and so are not real numbers.

The following is the last few lines of the above, but in the original Latin of the *Arithmetica Integra*, as brought by Tropfke*.

> "Sicut igitur infinitus numerus non est numerus,
> sic irrationalis numerus non est verus numerus,
> et latet sub quadam infinitatis nebula...Deinde,
> si numeri irrationalis essent numeri veri, tunc
> aut essent integri aut essent fracti...

---

* Tropfke J., *Geschichte der Elementarmathematik*, vol. I
Verlag Von Veit & Co., Leipzig 1902, p. 161.

b) Simon Stevin (1548-1620) was an engineer from the Netherlands. He made many important contributions in the field of statics and hydrostatics, which were particularly relevant in the Netherlands. But he also secured himself a place in the history of mathematics, because of his development of decimal fractions. The following short extract is taken from his book *Arithmetique* (1585), as it is quoted by Klein*.

**

"La partie est de la mesme matiere que son entier; Racine de 8 est partie de son quarré 8: Doncques $\sqrt{8}$ est de la mesme matiere que 8: Mais la matiere de 8 est nombre; Doncques la matiere de $\sqrt{8}$ est nombre: Et par consequent $\sqrt{8}$ est nombre."

---

* Klein J., *Greek Mathematical Thought and the Origin of Algebra*. M.I.T. Press, 1968, p. 290.

** This picture was taken from
Beck A., Bleicher M.N. & Crowe D.W., *Excursions into Mathematics*. Worth Pub. Inc., 1976, p. 446.

c) Albert Girard (1595-1632) was a student of Stevin and also from the Netherlands. Girard published Stevin's books, and his works include *Invention Nouvelle en L'Algebre* (1629). The following extract is taken from this book as quoted by Klein*.

> "Notez qu'on appelle *un nombre* tant les radicaux simples, comme est $\sqrt{2}$, ou $\sqrt{5071}$, que les multinomes, commes les binomes $2+\sqrt{5}$, item $7-\sqrt{48}$, item $\sqrt{26}-5$, comme les trinomes $4+\sqrt{2}-\sqrt{17}$, et autres multinomes, car ce qui lié par les signes soit+soit−ne font qu'un nombre."

Questions

1. What do you think Stifel meant when he wrote "when rational numbers fail us"?
2. What are Stifel's arguments for and against the acceptance of irrational numbers, and what is his conclusion?
3. Discuss Stevin's approach to roots and Girard's extension of this.
4. What does Girard mean by the term *multinome*, and what does this term mean today? What would we call Girard's *multinome*?
5. Could one of Girard's *multinomes* be a rational number?

---

* Ibid.

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The Irrational Numbers

Answer sheet: Irrationals in the 16th and 17th century.

1. It would seem reasonable to assume that Stifel was referring to finding the diagonal, given the side of the square. Or it could be the problem of finding the length of the side, given the area of a square. Etc.

2. Stifel's arguments are

For the acceptance of irrationals	against
- In geometry "irrational numbers... prove exactly those things which rational numbers could not prove".	- "they flee away perpetually", "...lies hidden in a kind of cloud of infinity"-meaning that they cannot be expressed
- "... the results which follow from their use-results which we perceive to be real, certain and constant."	

The mathematical historian Tropfke* adds that Stifel gives the irrational numbers a place in the sequence of numbers. Tropfke quotes the following extract from the *Arithmetica Integra* in the original Latin and with a German translation.

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* Tropfke, J., *Geschichte der Elementarmathematik*, Vol. 1, Von Veit and Co., Leipzig 1902, p. 166-167.

Item licet infiniti numeri fracti cadant inter quoslibet duos numeros immediatos, quemadmodum etiam infiniti numeri irrationales cadunt inter duos numeros integros immediatos. Ebenso wie die unbestimmten Brüche zwischen zwei benachbarte ganze Zahlen fallen, so fallen auch die unbestiminten Irrationalzahlen zwischen zwei benachbarte ganze Zahlen.

In spite of this, Stifel seems to conclude that "an irrational number is not a true number". Kline* states that this conclusion did not prevent Stifel from dealing with irrational numbers.

There were other contemporary mathematicians who worried about irrational numbers. Among them was J. Peletier (1517-1582), who wrote a chapter** entitled "Are the irrational numbers numbers or not, and of what kind?" The following extract is taken from the book by Klein,*** which gives the original Latin text with English translation.

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* Kline, M., *Mathematical Thought from Ancient to Modern Times*. O.U.P., 1972, p. 251.

** In a Latin text - *De Occulta Parte numerorum, quam Algebram Vocant (1560)*.

*** Klein, J., *Greek Mathematical Thought and the Origin of Algebra*, M.I.T. Press, 1968, p. 290.

"id tanquam in perpetuis tenebris delitescit" (this must lie hidden, as it were, in perpetual darkness). They are, in any case, "something," and it is certain that their *use is necessary* (necessarium usum), especially "in laying off continuous magnitudes" (praesertim in Continuorum dimensionibus). They are, just like genuine "absolute" numbers, subject to "rules" (praeceptiones). Their relation to the "absolute" numbers may be compared with that of the animals to man: "Habent igitur numeri Irrationales cum Absolutis obscuram quandam mutemque communicationem, non secus quam cum hominibus, Bruta: quae praeter id quod sentiunt, suo etiam modo ratiocinantur." (Irrational numbers, then, have a certain obscure and mute communication with absolute numbers, not differently from that which brutes, who besides having sense impressions, do, in their own way, even reason, have with men.) All in all, they are inexplicable (inexplicabiles) and have only a kind of shadow existence. They must not be counted among the numbers,

Peletier's arguments in favour of accepting the irrationals are similar to those of Stifel - "their use is necessary" - but he also is not willing to give them equal status with other numbers.

3. In the extracts, Stevin regards a root "as part of" its square in a sort of physical analogy - as if it were, for example a piece of metal. Every part of that metal is, of course, also metal - so a part of a number is a number. Whereas, Stifel took the mathematical view of a number as a sequence of digits, and got into difficulty with an infinite sequence, Stevin sidesteps the issue by using an implied analogy.
Girard states that roots are numbers - probably having accepted the arguments of his predecessors - and adds that arithmetic expressions obtained by adding or subtracting numbers are also numbers.

Note that the extracts are extremely short, and we should not come to very definite conclusions without examining them more fully in context. This, however, is not relevant here, where all we are trying to do is give an impression of the reaction to irrational numbers in various periods leading up to their final acceptance and definition.

4. Girard's *multinomes* are numerical (or arithmetical) expressions of the form $2 + \sqrt{5}$ or $4 + \sqrt{2} - \sqrt{17}$. *Multinome* today is not used - but would seem to be equivalent to polynome, which is an algebraic expression involving the addition (or subtraction) of a number of terms, usually positive integer powers of one or more variables.

5. Yes. For example $\sqrt{2} - \sqrt{2} = 0$, or

$$2 + \sqrt{27} - \sqrt{12} - \sqrt{3} = 2$$

But, if we take two different rationals, a and b, such that $\sqrt{a}$ and $\sqrt{b}$ are irrational , we can prove that $\sqrt{a} \pm \sqrt{b}$ is irrational.

For, suppose that $\sqrt{a} \pm \sqrt{b}$ is rational, i.e. that

$$\sqrt{a} \pm \sqrt{b} = \frac{m}{n}$$

Then $$\sqrt{a} = \frac{m}{n} \mp \sqrt{b}$$

Hence $$a = \left(\frac{m}{n}\right)^2 \mp \frac{2m}{n}\sqrt{b} + b$$

or $$\sqrt{b} = \mp \frac{n}{2m}\left[\, a - b - \left(\frac{m}{n}\right)^2 \,\right]$$

Now the right-hand side of this equation is rational, which contradicts the fact that $\sqrt{b}$ is irrational.

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The Irrational Numbers

Worksheet: Rafael Bombelli

L'ALGEBRA
OPERA
Di RAFAEL BOMBELLI da Bologna
Diuiſa in tre Libri.
Con la quale ciaſcuno da ſe potrà venire in perfetta cognitione della teorica dell'Arimetica.
Con vna Tauola copioſa delle materie, che in eſſa ſi contengono.
Poſta hora in luce à beneficio delli ſtudioſi di detta profeſſione.

IN BOLOGNA,
Per Giouanni Roſſi. MDLXXIX.
Con licenza de' Superiori.

A method of approximating roots of natural numbers

One of the problems which interested mathematicians was, how to approximate an irrational number which occurred as the root of a natural number which is not a perfect square. In this worksheet we will learn about a method for approximating roots, as presented by Bombelli in his book *L'Algebra Opera* (1579). Bombelli lived in the sixteenth century in Bologna. Very little about him is given in the history books*. However most of the histories of mathematics regard his book highly, as one of the most advanced of its time. This judgement was apparently based mainly on the first three parts, since the fourth and fifth part was first published in 1929, from an apparently unknown manuscript.

The following extract is taken from pp. 35-37 of the 1579 edition. We did not find one English translation to the whole original extract which is the object of this work-sheet. Nevertheless we include:

- the English translation to the first paragraph, by V. Sanford (in Smith D.E., *A Source Book in Mathematics*, McGraw Hill, N.Y., 1929, p. 80-82)
- the English translation of the last paragraph as it appears in Dedron, P. & Ytard, J. *Mathematics and Mathematicians*, Vol. 2, Transworld Pub., 1974, p. 69-70.

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* Details can be found, for example, in
Jayawardene S.A., Rafael Bombelli, Engineer Architect: Some Unpublished Documents..., *Isis*, 56, 1965. p. 298-306.

Modo di formare il rotto nella estrattione delle Radici quadrate.

Molti modi sono stati scritti da gli altri autori de l'vso di formare il rotto; l'uno tassando, e accusando l'altro (al mio giudicio) senza alcun proposito, perche tutti mirano ad un fine; E ben vero che l'una è più breue dell'altra, ma basta che tutte suppliscono, e quella ch'è più facile, non è dubbio ch'essa sarà accettata da gli huomini, e sarà posta in uso senza tassare alcuno; perche potria essere, che hoggi io insegnassi una regola, laquale piacerebbe più dell'altre date per il passato, e poi venisse un'altro, e ne trouasse una più vaga, e facile, e cosi sarebbe all'hora quella accetata, e la mia confutata, perche (come si dice) la esperienza ci è maestra, e l'opra loda l'artefice. Però metterò quella che più à me piace per hora, e sarà in arbitrio de gli huomini pigliare qual vorranno: dunque venendo al fatto dico. Che presupposto, che si voglia il prossimo lato di 13, che sarà 3, e auanzerà 4, il quale si partirà per 6 (doppio del 3 sudetto) ne uiene $\frac{2}{3}$, e questo è il primo rotto, che si hà da giongere al 3, che fà $3\ \frac{2}{3}$, ch'è il prossimo lato di 13, perche il suo quadrato è $13\ \frac{4}{9}$, ch'è superfluo $\frac{4}{9}$, ma uolendosi più approssimare, al 6. doppio del 3 se gli aggionga il rotto, cioè li $\frac{2}{3}$, e farà $6\ \frac{2}{3}$, e per esso partendosi il 4, che auanza dal 9 fino al 13, ne uiene $\frac{3}{9}$, e questo si gionge al 3, che fà $3\ \frac{3}{9}$, ch'è il lato prossimo di 13, di cui il quadrato è $12\ \frac{6}{9}\ \frac{4}{9}$, ch'è più prossimo di $3\ \frac{2}{3}$, ma uolendo più prossimo, si aggionga il rotto al 6 fa $6\ \frac{3}{9}$, e con esso si parta pur il 4, ne uiene $\frac{2}{3}\ \frac{6}{9}$, e questo si aggionga, come si è fatto di sopra al 3 fà $3\ \frac{2}{3}\ \frac{6}{9}$, ch'è l'altro numero più prossimo, perche il suo quadrato è $13\frac{}{1}.\frac{4}{9}\ \frac{4}{9}$, ch'è troppo $\frac{}{1}\frac{4}{9}\frac{4}{9}$, e uolendo più prossimo, partasi 4 per $6\ \frac{2}{3}\ \frac{6}{9}$, ne uiene $\frac{}{1}\ \frac{6}{9}\ \frac{4}{9}$, che gionto con il 3 fà $3\ \frac{}{1}\ \frac{6}{9}\ \frac{4}{9}$, e questo è più prossimo del passato, che il suo quadrato è $13\ \frac{}{1}.\frac{}{9}.\frac{}{9}.\frac{}{9}\ \ldots$ e cosi procedendo si può approssimare à una cosa insensibile.

Method of Forming Fractions in the Extraction of Roots

Many methods of forming fractions have been given in the works of other authors; the one attacking and accusing another without due cause (in my opinion) for they are all looking to the same end. It is indeed true that one method may be briefer than another, but it is enough that all are at hand and the one that is the most easy will without doubt be accepted by men and be put in use without casting aspersions on another method. Thus it may happen that today I may teach a rule which may be more acceptable than those given in the past, but if another should be discovered later and if one of them should be found to be more vague and if another should be found to be more easy, this [latter] would then be accepted at once and mine would be discarded; for as the saying goes, experience is our master and the result praises the workman. In short, I shall set forth the method which is the most pleasing to me today and it will rest in men's judgment to appraise what they see: meanwhile I shall continue my discourse going now to the discussion itself.

Let us first assume that if we wish to find the approximate root[1] of 13 that this will be 3 with 4 left over. This remainder should be divided by 6 (double the 3 given above) which gives $\frac{2}{3}$. This is the first fraction which is to be added to the 3, making $3\frac{2}{3}$ which is the approximate root of 13. Since the square of this number is $13\frac{4}{9}$, it is $\frac{4}{9}$ too large, and if one wishes a closer approximation, the 6 which is the double of the 3 should be added to the fraction $\frac{2}{3}$, giving $6\frac{2}{3}$, and this number should be divided into the 4 which is the difference between 13 and 9. The result is $\frac{3}{5}$ which, added to the 3 makes $3\frac{3}{5}$. This is a closer approximation to the root of 13, for its square is $12\frac{24}{25}$, which is closer than that of the $3\frac{2}{3}$. But if I wish a closer approximation, I add this fraction to the 6 making $6\frac{3}{5}$, divide 4 by this, obtaining $\frac{20}{33}$. This should be added to the 3 as was done above, making $3\frac{20}{33}$. This is a closer approximation for its square is $13\frac{4}{1089}$, which is $\frac{4}{1089}$ too large. If I wish a closer approximation, I divide 4 by $6\frac{20}{33}$, obtaining $\frac{66}{109}$, [and] add this to 3, obtaining $3\frac{66}{109}$. This is much closer than before for its square is $13\frac{96}{11881}$, which is $\frac{96}{11881}$ too large. If I wish to continue this even further, I divide 4 by $6\frac{66}{109}$ obtaining $\frac{109}{180}$, [the author now continues the process and then remarks:] and this process may be carried to within an imperceptible difference.

... epro
cedendo (come si è fatto di sopra) si approssimarà quanto l'huomo vorrà, e se bene ci sono molte altre regole: queste nondimeno mi sono parse le più facili, però a queste mi atterrò, le quali hò trouato con fondamento, qual non uoglio restare di porlo, benche non sarà intelo, se non da chi intende l'agguagliare di potenze, e tanti eguali à numeri, del quale tratterò nel secondo libro à pieno: Però hora parlo solo con quelli.

Pongasi dunque, che si habbia à trouare il lato prossimo di 13, di cui il più prossimo quadrato è 9; di cui il lato è 3, però pongo che il lato prossimo di 13. sia 3. p. 1 tanto, e il suo quadrato è 9. più 6 tanti p. 1. potenza', ilqual'è eguale à 13. che leuato 9. a ciascuna delle parti, resta 4, eguale à 6 tanti più 1 potenza. Molti hanno lasciato andare quella potenza, e solo hanno agguagliato 6 tanti à 4, che il tanto valeria $\frac{2}{3}$ & hanno fatto, che l'approssimatione si è 3 $\frac{2}{3}$ perche la positione fù 3. p. 1. tanto, uiene ad essere 3 $\frac{2}{3}$, ma uolendo tenere conto della potenza ancora, valendo il tanto $\frac{2}{3}$, la potenza ualerà $\frac{2}{3}$ di tanto, che aggionto con li 6 tanti di prima: si hauerà 6 $\frac{2}{3}$ tanti eguale à 4, che agguagliato il tanto valerà $\frac{2}{3}$, e perche fù posto 3. p. 1. tanto, sarà 3 $\frac{2}{3}$, e ualendo il tanto $\frac{2}{3}$, la potenza valerà $\frac{2}{3}$ di tanto, e si hauerà 6 $\frac{2}{3}$ di tanto eguale à 4, si che si uede donde nascono le regole dette di sopra.

Let us suppose we are required to find the root of 13. The nearest square is 9, which has root 3. I let the approximate root of 13 be 3 plus 1 tanto. Its square is 9 plus 6 tanti p. 1 power. We set this equal to 13. Subtracting 9 from either side of the equation we are left with 4 equal to 6 tanti plus 1 power.

Many people have neglected the power and merely set 6 tanti equal to 4. The tanto then comes to $\frac{2}{3}$ and the approximate value of the root is $3\frac{2}{3}$ since it has been set equal to 3 p. 1 tanto. However, taking the power into account, if the tanto is equal to $\frac{2}{3}$, the power will be $\frac{2}{3}$ of a tanto, which, added to the 6 tanti, will give us $6\frac{2}{3}$ tanti, which are equal to 4. So the tanto will be equal to $\frac{1}{3}$, and since the approximate root is 3 p. 1 tanto it comes to $3\frac{1}{3}$. But if the tanto is equal to $\frac{1}{3}$, the power will be $\frac{1}{3}$ of a tanto and we obtain $6\frac{1}{3}$ tanti equal to 4....

Questions

1. At the end of the extract, Bombelli gives the principles of his method for *"chi intende l'agguagliare di potenze, e tanti equali à numeri"* (those who understand how to equalize powers and unknowns numbers).

   Complete the following "dictionary".

Bombelli	English translation	Modern notation
p. or *piu*	plus	------
equale	equal	------
I tanto	one quantity (unknown)	------
potenza	power (second, of unknown)	------
3 p. I tanto	------	3 + x
9. piu 6 tanti p. I potenza	------	------

2. The stages in Bombelli's method of approximating $\sqrt{13}$ are set out in the following table. Complete the blanks

Bombelli (original)	English version	Modern Notation	The method
		$(3+x)^2=9+6x+x^2=13$	
... pongo che il lato proſſimo di 13. ſia 3. p. 1 tanto, e il ſuo quadrato è 9. piu 6 tanti p. 1. potenza', ilqual'è eguale à 13. che leuato 9. a ciaſcuna delle parti, reſta 4, eguale à 6 tanti più 1 potenza. Molti hanno laſciato andare quella potenza, e ſolo hanno agguagliato 6 tanti à 4, che il tanto valeria $\frac{2}{3}$ & hanno fatto, che l'approſſimatione ſi è $3\frac{2}{3}$...	I.e. the approximate root of 13 be 3 plus 1 tanto. Its square is 9 plus 6 tanti p. 1 power. We put this equal to 13. Subtracting 9 from either side of the equation we are left with 4 equal to 6 tanti plus 1 power. Many people have neglected the power and merely set 6 tanti equal to 4. The tanto then comes to $\frac{2}{3}$ and the approximate value of the root is $3\frac{2}{3}$...	---------- ---------- ---------- ----------	1st. approx. x_1: $6x_1=4$ $x_1=\frac{4}{6}$
... ma co-lendo tenere conto della potenza ancora, valendo il tanto $\frac{2}{3}$, la potenza ualerà $\frac{2}{3}$ di tanto, che aggionto con li 6 tanti di prima: ſi hauerà $6\frac{2}{3}$ tanti eguale à 4, che agguagliato il tanto valerà $\frac{2}{3}$...	However, taking the power into account, if the tanto is equal to $\frac{2}{3}$, the power will be $\frac{2}{3}$ of a tanto, which, added to the 6 tanti, will give us $6\frac{2}{3}$ tanti, which are equal to 4. So the tanto will be equal to $\frac{2}{3}$...	$6x+x^2=4$ $6x+xx=4$ ------ ------	2nd approx. x_2: $6x_2+\frac{4}{6}x_2=4$ ----------
... e ſi hauerà $6\frac{2}{3}$ di tanto eguale à 4, ...	...and we obtain $6\frac{2}{3}$ tanti equal to 4....	------	3rd approx. x_3: ---------- ----------

3. In the section where Bombelli approximates $\sqrt{13}$ (on page 2), there is a mistake. Find it! What is its probable cause? (This question refers to the original Italian because the mistake does not appear in the English version.)

4. Approximate $\sqrt{2}$ by Bombelli's method, accurate to 4 decimal places (1.4142).

5. a) If b is not a perfect square, write down the iteration formula which gives the approximation to $\sqrt{b}$ by Bombelli's method.

   b) Is the first approximation to $\sqrt{b}$, greater than or less than $\sqrt{b}$?

   c) What about the second and third approximation?

   d) Generalise !

6. Bombelli writes "*... e cosi procedendo si puo approssimare a una cosa insensibile*" (... and this process may be carried to within an imperceptible difference), but in the extract, he "omits" the fundamental mathematical question, whether, if we continue the process indefinitely, one can prove that the sequence of approximations converges to the desired root. Do it in two stages:

   a) Show that the odd approximations are steadily decreasing and the even ones are increasing.
   (Remember also what you discovered in the previous question.)

   b) Show that the odd and the even approximations converge to the desired root.

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The Irrational Numbers

Answer sheet: Rafael Bombelli

A method for approximating roots of natural numbers

1. Bombelli	English translation	Modern notation
p. or *piu*	plus	+
equale	equal	=
I tanto	one quantity (unknown)	x
potenza	power (second, of unknown)	x^2
3 p. I tanto	3 plus one quantity	3 + x
9 piu 6 tanti p. I potenza	9 plus 6 quantities plus one power	$9 + 6x + x^2$

Bombelli (original)	English version	Modern Notation	The method
... pongo che il lato proſſimo di r 3. ſia 3. p. 1 tanto, e il ſuo quadrato è 9. piu 6 tanti p. 1. potenza', ilqual è eguale à 13. che leuato 9. a ciaſcuna delle parti, reſta 4, eguale à 6 tanti più 1 potenza. Molti hanno laſciato andare quella potenza, e ſolo hanno agguagliato 6 tanti à 4, che il tanto valeria $\frac{4}{6}$ & hanno fatto, che l'approſſimatione ſi è $3\frac{4}{6}$...	1 is the approximate root of 13 be 3 plus 1 tanto. Its square is 9 plus 6 tanti p. 1 power. We put this equal to 13. Subtracting 9 from either side of the equation we are left with 4 equal to 6 tanti plus 1 power. Many people have neglected the power and merely set 6 tanti equal to 4. The tanto then comes to $\frac{4}{6}$ and the approximate value of the root is $3\frac{4}{6}$...	$(3+x)^2=9+6x+x^2=13$ $9+6x+x^2-9=13-9$ $6x+x^2=4$ $\vdots$ $6x=4$ $x=\frac{2}{3}$ $3+\frac{2}{3}=3\frac{2}{3}$	1st. approx. x_1: $6x_1=4$ $x_1=\frac{4}{6}$
... ma uolendo tenere conto della potenza ancora, valendo il tanto $\frac{4}{6}$, la potenza ualerà $\frac{4}{6}$ di tanto, che aggionto con li 6 tanti di prima: ſi hauerà $6\frac{4}{6}$ tanti eguale à 4, che agguagliato il tanto valerà $\frac{4}{6}$...	However, taking the power into account, if the tanto is equal to $\frac{4}{6}$, the power will be $\frac{4}{6}$ of a tanto, which, added to the 6 tanti, will give us $6\frac{4}{6}$ tanti, which are equal to 4. So the tanto will be equal to $\frac{4}{6}$...	$6x+x^2=4$ $6x+x_1x=4$ $6x+\frac{2}{3}x=4$ $6\frac{2}{3}x=4$ $x=\frac{3}{5}$	2nd approx. x_2: $6x_2+\frac{4}{6}x_2=4$ $x_2=\frac{4}{6+\frac{4}{6}}$
... e ſi hauerà $6\frac{4}{6}$ di tanto eguale à 4, ...	... and we obtain $6\frac{4}{6}$ tanti equal to 4....	$6\frac{3}{5}x=4$	3rd approx. x_3: $6x_3+\frac{4}{6+\frac{4}{6}}x_3=4$ $x_3=\frac{4}{6+\frac{4}{6+\frac{4}{6}}}$

Thus, we can write the general procedure in the form

$$\sqrt{13} = 3 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + \ddots}}}$$

This expression is today known as an *infinite continued fraction* - not that Bombelli wrote it this way or called it by this name. Nevertheless, he is credited with being one of the first to develop the subject, which has an interesting history of its own.

3. After the approximation $3\frac{20}{33}$, Bombelli writes:

> uolendo piu proſſimo, partaſi 4 per 6 $\frac{\cdot}{\cdot}$ $\frac{\cdot}{\cdot}$, ne uie- ...e
> ne $\frac{\cdot}{\cdot}$ $\frac{\cdot}{\cdot}$ $\frac{\cdot}{\cdot}$, che gionto con il 3 fà 3 $\frac{\cdot}{\cdot}$ $\frac{\cdot}{\cdot}$ $\frac{\cdot}{\cdot}$...

but $4 \div 6\frac{20}{33} = 4 \times \frac{33}{218} = \frac{66}{109}$,

and not $\frac{109}{180}$, as appears in the original.

If we continue the process, we get

$$4 \div 6\frac{66}{109} = 4 \times \frac{109}{720} = \frac{109}{180}.$$

So, it would seem that the error occurred as a result of skipping one step, maybe caused by missing out a line from the manuscript when setting the type. (It is interesting to note that in Smith* this section is given in English translation, for the same Italian edition as we have used here, without any error.)

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* Smith, D.E. *A Source Book in Mathematics*, McGraw-Hill, 1929, p. 80-82.

4. $(1 + x)^2 = 1 + 2x + x^2 = 2$

$$2x + x^2 = 1$$

If we ignore the x^2, we obtain

$$x_1 = \frac{1}{2}$$

Hence the first approximation is $1\frac{1}{2}$, whose square is $2\frac{1}{4}$.

Continuing the procedure, we obtain

$$2x_2 + \frac{1}{2}x_2 = 1$$

$$2\frac{1}{2}x_2 = 1$$

$$x_2 = \frac{1}{2\frac{1}{2}} = \frac{2}{5}$$

Hence, the second approximation is $1\frac{2}{5}$, whose square is $1\frac{24}{25}$.

At the next stage

$$2x_3 + \frac{2}{5}x_3 = 1$$

$$x_3 = \frac{1}{2\frac{2}{5}} = \frac{5}{12}$$

Hence the third approximation is $1\frac{5}{12}$ $(1.41\dot{6})$

Etc.

5. a) Let a be the largest natural number whose square is less than b, that is

$$a^2 < b < (a + 1)^2.$$

Hence, there exists an x, $0 < x < 1$, such that

$$(a + x)^2 = b$$

$$a^2 + 2ax + x^2 = b$$

$$2ax + x^2 = b - a^2$$

For the first approximation, we ignore x^2 (since $x < 1$, x^2 is small in comparison with $2ax$) and obtain

$$2ax_1 = b - a^2$$

$$x_1 = \frac{b - a^2}{2a}$$

For the second approximation, we write

$$2ax_2 + x_1x_2 = b - a^2$$

$$(2a + x_1)x_2 = b - a^2$$

$$x_2 = \frac{b - a^2}{2a+x_1}$$

In general

$$x_{n+1} = \frac{b - a^2}{2a + x_n}$$

The nth approximation to $\sqrt{b}$ is then $a + x_n$.

b) From the previous section, the first approximation to $\sqrt{b}$ is $a + x_1$.

$$(a + x_1)^2 = a^2 + 2ax_1 + x_1^2$$

But since $a^2 + 2ax_1 = b$, we have

$$(a + x_1)^2 = b + x_1^2 > b .$$

Thus the first approximation is greater than $\sqrt{b}$.

c) $$x_2 = \frac{b - a^2}{2a + x_1}$$

Hence $(2a + x_1)x_2 = b - a^2$

$$[(a + x_1) + a]x_2 = (\sqrt{b} - a)(\sqrt{b} + a)$$

But $a + x_1$ is the first approximation, and we have shown in the previous section that $a + x_1 > \sqrt{b}$. Hence replacing $a + x_1$,by $\sqrt{b}$, we have

$$(\sqrt{b} + a)x_2 < (\sqrt{b} - a)(\sqrt{b} + a)$$

i.e. $$x_2 < \sqrt{b} - a$$

or $$a + x_2 < \sqrt{b}$$

showing that the second approximation is less than $\sqrt{b}$.

We can show similarly that the third approximation is again greater than $\sqrt{b}$.

d) Exactly the same sort of argument will show that every even approximation is always less than b, and every odd approximation always greater. In symbols

$$a + x_{2n} < \sqrt{b}$$

$$a + x_{2n-1} > \sqrt{b} \qquad n \in N$$

6. a) We have to show that for $n \in N$

$$x_{2n-1} > x_{2n+1}$$

or $$x_{2n-1} - x_{2n+1} > 0 .$$

Using the iteration formula we found in the previous question, we have

$$x_{2n+1} = \frac{b - a^2}{2a + x_{2n}}$$

$$x_{2n+1} = \frac{b - a^2}{2a + \dfrac{b - a^2}{2a + x_{2n-1}}}$$

$$x_{2n+1} = \frac{(b-a^2)\cdot(2a+x_{2n-1})}{2a\cdot(2a+x_{2n-1}) + b - a^2}$$

hence

$$x_{2n-1} - x_{2n+1} = x_{2n-1} - \frac{(b-a^2)\cdot(2a+x_{2n-1})}{2a(2a+x_{2n-1}) + b - a^2}$$

$$= \frac{x_{2n-1}[2a(2a+x_{2n-1}) + b - a^2] - (2a + x_{2n-1})\cdot(b-a^2)}{2a(2a+x_{2n-1}) + b - a^2}$$

Since the denominator is positive, we need only examine the numerator, which becomes

$$x_{2n-1} \cdot 2a \cdot (2a + x_{2n-1}) - 2a(b - a^2)$$

$$= 2a[x_{2n-1}(2a + x_{2n-1}) - (b - a^2)]$$

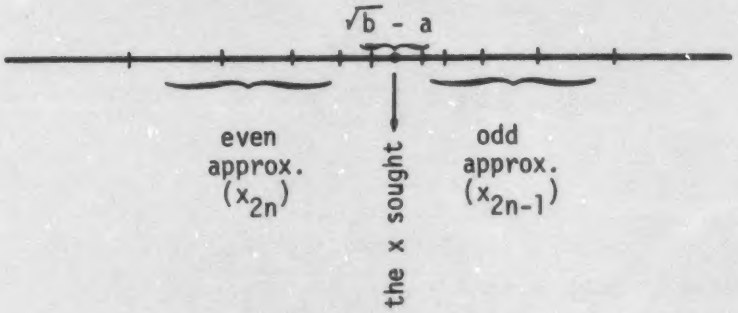
$$= 2a[x_{2n-1}^2 + 2ax_{2n-1} + a^2 - b]$$

$$= 2a[(x_{2n-1} + a)^2 - b] .$$

Since odd approximations are greater than $\sqrt{b}$, the expression in brackets is positive; i.e. $x_{2n-1} > x_{2n+1}$

Similarly, it can be shown that each even approximation is greater than its predecessor.

b) From what we have proved so far, the situation is described by the following graph



The odd approximations are decreasing but are always greater than $\sqrt{b} - a$. Therefore, they must approach some limit, ℓ_1 say, such that

$$\ell_1 > \sqrt{b} - a$$

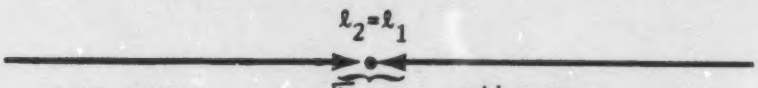
Similarly the even approximations approach some limit ℓ_2 , such that

$$\ell_2 < \sqrt{b} - a$$

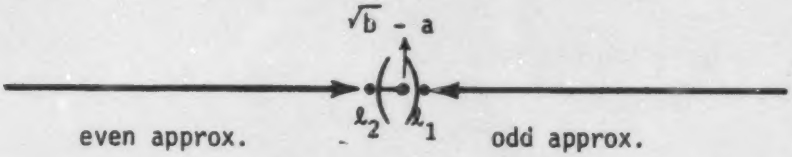
It remains to show that

$$\ell_1 = \ell_2 = \sqrt{b} - a \ .$$

That is



and that we do not create a situation such as, for example



Now, we have

$$x_n = \frac{b - a^2}{2a + x_{n-1}} = \frac{b - a^2}{2a + \dfrac{b - a^2}{2a + x_{n-2}}}$$

If n is even (odd), then n-2 is even (odd). As n increases, x_n and x_{n-2} are very close to ℓ_2 (ℓ_1). Hence

$$\ell_1 = \frac{b - a^2}{2a + \frac{b - a^2}{2a + \ell_1}}$$

$$\ell_1 = \frac{(b - a^2)\cdot(2a+\ell_1)}{2a(2a+\ell_1) + b - a^2}$$

$$[2a(2a+\ell_1) + b - a^2]\ell_1 = (b-a^2)\cdot(2a+\ell_1)$$

$$2a\ell_1(2a+\ell_1) = 2a(b-a^2)$$

$$\ell_1^2 + 2a\ell_1 + a^2 = b$$

$$(\ell_1 + a)^2 = b$$

whence $$\ell_1 = x$$

Similarly one can show that

$$\ell_2 = x .$$

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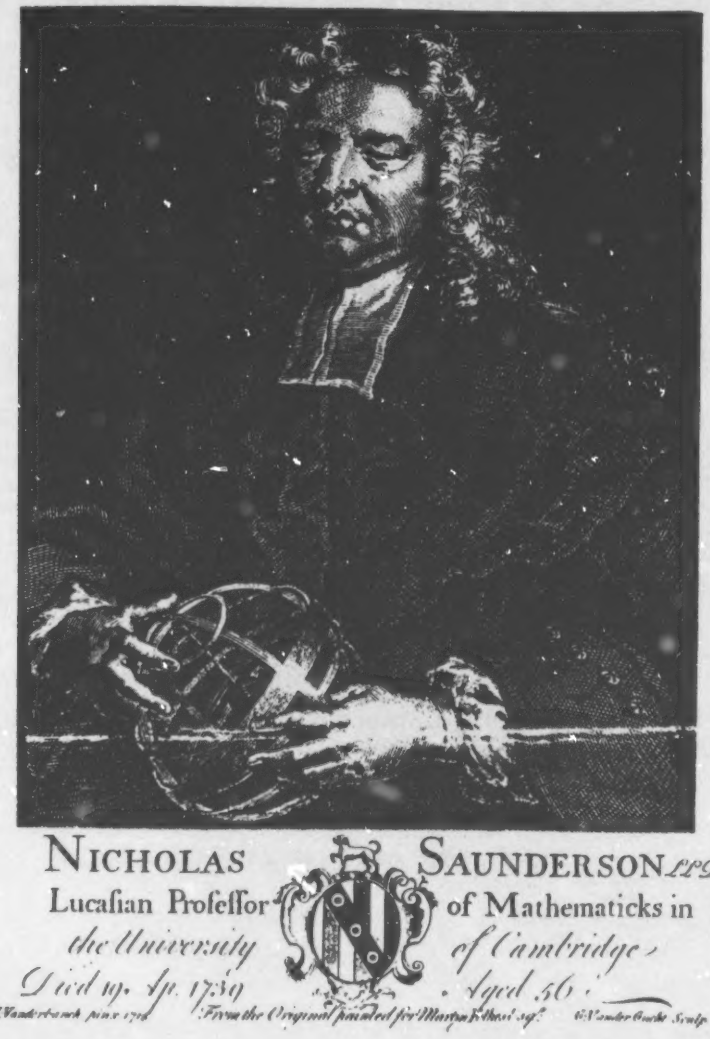
The Irrational Numbers

Worksheet: Nicholas Saunderson



Calculating the square root

Nicholas Saunderson (1682-1739), an English mathematician, became blind at the age of one as a result of smallpox. This did not prevent him from learning, teaching and creating mathematics. As a young man he learnt Greek, Latin and French, and hence "heard" the works of Euclid, Archimedes, Diophantes and others in the original, and remembered parts of them by heart.

His father taught him arithmetic and he showed considerable ability in solving problems and long calculations, using his prodiguous memory and a "geoboard", which he developed. The "geoboard" was a pegboard with arrays of nails which represented numbers. By passing silk threads around the nails, he could "draw" geometrical figures.

Saunderson continued his mathematical studies with great success, having texts read to him. He liked teaching mathematics, and gave very clear expositions. At Cambridge he had the distinction of occupying the chair held by Newton some years earlier.

The following extracts are taken from his book The *Elements of Algebra** (pages 165-9), which was published posthumously in 1741. In the extracts he gives a mathematical explanation of the "usual rule" for finding square roots.

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* Further details on the man and his work can be found in the introduction to the book itself,
Saunderson N., *The Elements of Algebra*, Vol 1. Cambridge Univ. Press, 1741.

For the better effecting what I here propoſe, I ſhall lay down the following obſervations, which the learner muſt attend to, if he expects to go through the following demonſtration; and if there be ſtill any difficulties he may meet with or thinks he meets with in the application of theſe obſervations, the beſt advice I can give him here, as well as in many other parts of this book, is to read the demonſtration over and over again, by which means all the ſteps will become more familiar to him, and he will be the better able to put them together, in order to digeſt the whole.

Observation 1.

If any number, as $a+x$, conſiſting of two parts a and x, be ſquared, the product will be $aa+2ax+xx=aa+\overline{2a+x}\times x$.

Observation 2.

$1\times1=1$; therefore $10\times10=100$, and $100\times100=10000$, and $1000\times1000=1000000$, *&c*: whence I infer, that if any number conſiſts of one or two places, it's ſquare root, or at leaſt the integral part of it, will conſiſt but of one place; if a number conſiſts of three or four places, the integral part of it's ſquare root will conſiſt of two places...

Observation 3.

From the laſt obſervation it follows, that if of any number propoſed, a point be put over the place of units, and another over the next place but one to the left hand, I mean that of hundreds, and another again over the next place but one to that, to wit, of ten thouſands, and ſo on alternately; the number of points will diſcover the number of places of which the integral part of the ſquare root of the propoſed number conſiſts: thus if the number 56644 be ſo pointed, it will ſtand thus, $\dot{5}6\dot{6}44$; and the three points ſhew, that the ſquare root of this number, or the integral part of it at leaſt, conſiſts of three places.

Observation 4.

...if the ſquare root of the number 576 be 24, that of the number 57600 will be 240...

Observation 5.

...if the ſquare root of any number, ſuppoſe of 5 .6644 lies between 2 and 3, the ſquare root of 566 .44 will lie between 20 and 30...

The inveſtigation of the rule for extracting the ſquare root.

Theſe things being obſerved, let now ſome number be propoſed, ſuch as is the number 56644; and let us ſee whether we cannot inveſtigate the ſquare root of this number by pure dint of reaſon, without any regard to, or helps from the common method.* This number then being pointed according to the third obſervation, I ſhall firſt begin with the number 5, belonging to the firſt point to the left hand, ſetting aſide all the other figures, thus: the number 5 is itſelf no ſquare number, therefore I ſubtract the number 4 which is the neareſt ſquare number leſs than 5, from 5, and there remains 1...

...the number 2 will be the firſt term of the ſquare root,...

...now to diſcover the next figure in the root, I conſider the number 566 belonging to the two firſt dotts to the left hand, ſetting aſide all the other figures,...

...the ſquare root of the number 566 lies between 20 and 30; let then $20+x$ repreſent the integral part of the ſquare root of the number 566, the letter x ſtanding for ſome whole number in the place of units; then it is plain, that the ſquare of $20+x$ muſt either be preciſely equal to 566 if 566 be an exact ſquare, or elſe it muſt be the neareſt ſquare number leſs than 566; but the ſquare of $20+x$ by the firſt obſervation is $20\times20+\overline{40+x}\times x$; therefore $20\times20+\overline{40+x}\times x$ is either equal to, or leſs than 566; ſubtract the ſquare of 20 from both ſides, by obſerving, that as before, 2×2 ſubtracted from 5 left 1, ſo now 20×20 ſubtracted from 500 will leave 100, and the ſame ſubtracted from 566, will leave 166; therefore $\overline{40+x}\times x$ muſt either be equal to, or leſs than 166... ...it muſt be reſolved into two factors, x and $40+x$,...

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* According to the introduction to this section (which we have not brought here), Saunderson means by the "common method", the method which he is about to explain. However, he asks the reader to forget the rules, in order to better understand their mathematical justification.

Questions

1. i) What is the significance of the line above $\overline{2a + x}$?

   ii) Write Observation 1 in modern form.

2. Generalise the second observation for the square root (or the integer part thereof) of an n-digit number.

3. Generalise the fourth observation for $a \cdot 10^n$, where $\sqrt{a} = b$.

4. Complete Saunderson's explanation of the finding of the square root of 56644.

5. Find $\sqrt{13}$, by the method explained in the extract, to three decimal places.

6. How does Saunderson's method differ from that of Bombelli?

7. Are you aware of any other methods for finding square roots?

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The Irrational Numbers

Answer sheet: Nicholas Saunderson

Calculating the square root

1. The line above $\overline{2a + x}$ is used where we today would use brackets. Thus we write $a^2 + (2a + x)x$ instead of $aa + \overline{2a + x} \times x$.

   Although the use of brackets to associate terms can be found in 16th century texts, according to Cajori*, they did not become widely used until the 18th century. Beside the line as used by Saunderson, terms to be associated were denoted by letters, commas or full stops. The line, which was called a *vinculum*, persisted in use, even after the general introduction of brackets. For example, the following expression is taken from a textbook edited in 1950**, in which both the *vinculum* and the brackets are used.

$$= \frac{1}{2}\left\{2an_2 + 2n_1n_2d - n_2(n_2 + 1)\,d\right\}$$

$$= \frac{1}{2}\left\{2n_1n_2d - (dn_2^2 - \overline{2a - d}\,.\,n_2)\right\}$$

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* Cajori F., *A History of Mathematical Notations*, Vol. I The Open Court Pub. Co. 1928, p. 384-400.

** Hall H.S. & Knight S.R., *Higher Algebra*, Macmillan & Co., London, 1950, p. 34.

In the end, it is likely that the brackets triumphed for reasons associated with the printing of mathematical texts, rather than mathematical convenience.

2. According to Observation 2, the integral part of the square root of each number between 1 and 100, consists of one digit, between 100 and 10,000 of two digits, etc. Thus if a number consists of n digits, then if

   n is <u>even</u>, the integral part of the square root will have $\frac{n}{2}$ digits,

   n is <u>odd</u>, the integral part of the square root will have $\frac{n+1}{2}$ digits.

3. If $\sqrt{a} = b$ and n is even, then $\sqrt{a \cdot 10^n} = b\,10^{\frac{n}{2}}$.

4. In Saunderson, the continuation is as follows.

> ... the product of whoſe multiplication muſt either be equal to 166, or elſe it muſt be the neareſt product of the kind leſs: but to proceed; as it has been ſhewn already that the product $\overline{40+x}\times x$ is not to exceed 166, it follows, that the product $40x$ muſt be leſs than 166, and conſequently that x muſt be leſs than the quotient of 166 divided by 40, or of 16.6 divided by 4; but the quotient of 16.6 divided by 4 lies between the two whole numbers 4 and 5; therefore x is leſs than 5, and 4 is the greateſt whole number that can be ſuppoſed equal to x; let us then ſuppoſe x equal to 4, and let us ſee what will be the conſequence; now if $x=4$, we ſhall have $40+x=44$, and $\overline{40+x}\times x = 44\times 4 = 176$; therefore the ſuppoſition of $x=4$ was wrong, becauſe $\overline{40+x}\times x$ ought not to exceed 166; let us then ſuppoſe in the next place that $x=3$; then we ſhall have $40+x=43$, and $\overline{40+x}\times x = 43\times 3 = 129$, which is leſs than 166, and therefore not inconſiſtent with the foregoing conditions; and ſince x muſt be taken equal to the greateſt whole number the foregoing conditions will admit of, it follows, that x muſt be equal to 3, and conſequently that $20+x$, or

Art. 99, 100. *The foundation of the rule for extracting the cube root.* 169

or the integral part of the ſquare root of the number 566 is 23; and ſince $\overline{40+x}\times x$ or 129, ſubtracted from the reſolvend 166 leaves 37, it follows, that 23×23 ſubtracted from the number 566 will alſo leave 37: again, as the ſquare of 23 is the neareſt ſquare number leſs than 566 which is no ſquare, the ſquare of 24 muſt be greater than 566, and conſequently cannot be leſs than 567; therefore the ſquare root of the number 566.44 muſt neceſſarily lie between 23 and 24; therefore by the fifth obſervation, the ſquare root of the number 56644 muſt neceſſarily lie between 230 and 240; which is another ſtep made in the approximation: let us now ſuppoſe $230+x$ to be the ſquare root, or the integral part of the ſquare root of the number 56644; then by the firſt obſervation we ſhall have $230\times230+\overline{460+x}\times x$ either equal to, or leſs than 56644: ſubtract 230×230 from both ſides, by obſerving, that as before 23×23 ſubtracted from 566 left 37, ſo now 230×230 ſubtracted from 56600 will leave 3700; and the ſame ſubtracted from 56644 will leave 3744; therefore $\overline{460+x}\times x$ muſt not exceed 3744; therefore $460x$ muſt be leſs than 3744; therefore x muſt be leſs than the quotient of 3744 divided by 460; but the quotient of 3744 divided by 460, or (which is pretty much the ſame thing, eſpecially with reſpect to the integral part) the quotient of 374 divided by 46 lies between the two whole numbers 8 and 9; therefore 8 is the greateſt whole number that can be ſuppoſed equal to x: let us then ſuppoſe $x=8$, and we ſhall have $\overline{460+x}=468$, and $\overline{460+x}\times x=468\times8=3744$, which is juſt conſiſtent with the abovementioned conditions, and the number 238 is the exact ſquare root of the number propoſed 56644. *Q. E. I.*

We expect that your answer was somewhat shorter.

In order to organize the calculation, we suggest the following form.

$\sqrt{5\,6\,6\,4\,4}$ = 238

```
-
  4
-----
  166
-
  129
-----
   3744
-  3744
-------
   0000
```

$\boxed{2} \cdot 2 = 4$

$(40 + x) \cdot x \leq 166 \quad / x = 3$

$4\boxed{3} \cdot \boxed{3} = 129$

$23 \cdot 2 = 46$

$(460 + x) \cdot x \leq 3744 \quad / \quad x = 8$

$46\boxed{8} \cdot \boxed{8} = 3744$

5. $\sqrt{13.000000...}$ = 3.605...

```
-
  9
-----
  400
-
  396
-----
     400
-    0
-------
   40000
- 36025
-------
    3975
```

$\boxed{3} \cdot 2 = 6$

$(60 + x) \cdot x \leq 400 \quad / x = 6$

$6\boxed{6} \cdot \boxed{6} = 396$

$36 \cdot 2 = 72$

$(720 + x) \cdot x \leq 400 \quad / x = 0$

$720\boxed{0} \cdot \boxed{0} = 0$

$360 \cdot 2 = 720$

$(7200 + x) \cdot x \leq 40000 \quad / x = 5$

$720\boxed{5} \cdot \boxed{5} = 36025$

In the following passage, Saunderson describes how to deal with numbers which are not perfect squares.

There are but few ſquare numbers, or ſuch as will admit of an exact ſquare root, in compariſon of the reſt; and therefore, whenever a number is propoſed to have it's ſquare root extracted, the artiſt muſt firſt determine with himſelf, to how many decimal places it is proper the root ſhould be continued; and then by annexing decimal cyphers, if need be, to the right hand of the number propoſed, he muſt make twice as many decimal places there, as the root is to conſiſt of; after this, he muſt put a point over the place of units, and then paſſing by every other figure, he muſt point in like manner all the reſt, both to the right hand, and to the left: by this means, the number will be prepared,...

That the ſuppoſed ſquare ought to have twice as many decimal places as the root, is evident, both *à priori*, and *à poſteriori*: *à priori*, becauſe in extracting the ſquare root, two figures are brought down from the ſquare for every ſingle figure gained in the root; and *à poſteriori*, becauſe the root multiplied into itſelf is to produce the ſquare; and therefore, from the nature of multiplication, the ſquare ought to have twice as many decimal places as the root.

6.

a) The method is essentially to "find" the root of any number, including perfect squares

a) The method is essentially to find approximations to the square root of a whole number, given the largest integer whose square is less than the given number.

b) The values obtained in successive stages are always less than the desired square root (except in the case of a perfect square, in the final stage).

b) The values obtained in successive stages are alternately smaller and larger than the square root.

c) The digits found in the successive stages of the process, do not change in subsequent stages. For example,

First approx. $\sqrt{2} = 1.4$
Second approx. $\sqrt{2} = 1.41$
Third approx $\sqrt{2} = 1.414$
Fourth approx. $\sqrt{2} = 1.4142$

c) The digits found in successive stages of the process, may change in subsequent stages. For example,

First approx. $\sqrt{2} = 1.5$
Second approx. $\sqrt{2} = 1.4$
Third approx. $\sqrt{2} = 1.41\dot{6}$
Fourth approx. $\sqrt{2} = 1.413...$.

The method described by Saunderson has a long history. According to Smith*, in the fourth century, Theon of Alexandria found roots by this method, as did the Arabs, Hindus

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* Smith, D.E., *History of Mathematics*, Vol. II, Dover, 1958, pp. 144-9.

and other mathematicians in the Middle Ages and in more modern times. But it was only later that explanations were added to justify the method. The method was taught in schools until very modern times - in spite of tables and slide rules - finally to be displaced by the calculator.

7. a) See for example the following extract*.

General Method.

Graphic Illustration. Suppose the square ABCD represents the number, and that it is known that *Abcd*, representing say n^2, is very nearly equal to it, so that n is a sort of first guess to the square root of the number.

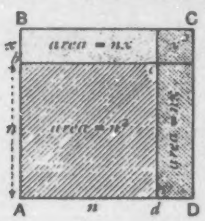

FIG. 37.

The problem is therefore to find the additional amount bB, representing say x, which has to be added to n to obtain the required square root.

From the figure it is clear that what is wanted is a number x, such that twice the product of x and n, together with the square of x, is equal to the difference between the square of n and the given number.

Further, that ABCD – *Abcd* is very nearly $2n \times x$, and therefore that x is *suggested by* (ABCD – *Abcd*) ÷ ($2n$).

This is the key of the Arithmetical Method of extracting square roots.

EXAMPLE.—*To evaluate* $\sqrt{4225}$.

Beginning with units and tens digits, divide up the number so far as possible into 'periods' of two digits.

		5 = Units } portion of $\sqrt{4225}$;
60	42|25	60 = Tens } ∵ 4225 lies between 60^2 and 70^2.
	36 00	Set down 60 on both sides; multiply them as shown.
Twice 60 = 120	6 25	... = $4225 - 60^2$; $4225 - (60)^2 \div 120$ suggests 5 as units digit.
Add the 5	5	
	125 | 6 25	Multiply 125 by 5.

Since there is no remainder $65 = \sqrt{4225}$.

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* Taken from: Jones H.S., *A Modern Arithmetic with Graphic & Practical Exercises*, Macmillan & Co., 1909, p. 274-5.

b) With the advent of the calculator, a reasonable method for finding the square root of any number (without using the $\sqrt{}$ key) is trial and error. This can be formulated in the form of a systematic algorithm, and used quite generally.

c) Heron of Alexandria lived about the first century B.C.E. He is credited* with the following method and its explanation (which is given here in modern form).

If a_1 is a first approximation to $\sqrt{A}$, the Heron's second approximation is

$$a_2 = \frac{1}{2}\left(a_1 + \frac{A}{a_1}\right)$$

and, in general,

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{A}{a_{n-1}}\right)$$

This method also became known as Newton's (or Newton-Raphson) method, because it can be derived as a special case of a more general method for finding the roots of equations.

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* Dedron, P. and Itard, J., *Mathematics and Mathematicians*, Vol. 2, Transworld pub., 1973, pp. 67-8.

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The Irrational Numbers

Worksheet: Dedekind and the definition of the irrationals.

Introduction

1. What is the nearest rational to $\frac{1}{2}$?
2. How many rationals are there between 0.7 and 0.71?
3. Are all the numbers in the interval [0.7, 0.71] rational?
4. What is the number nearest to and greater than $\sqrt{2}$, and the nearest number smaller than it?

J.W.R. Dedekind

Julius Wilhelm Richard Dedekind (1831-1916) was a German mathematician. He developed a formal arithmetic definition of the irrational numbers, assuming the rationals as given. Dedekind was one of a number of distinguished German mathematicians who concerned themselves with this topic in the same period. Others included G. Cantor (1845-1918) and Weierstrass (1815-1897). The following extracts are taken from the English version of one of Dedekind's papers*. The original *Stetigkeit un die Irrationalzahlen* (Continuity and Irrational Numbers) was first published in 1872. He describes how the idea was born, and then gives the definition.

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* Dedekind R., *Essays on the Theory of Numbers*. Open Court, 1924, pp. 1-24.

CONTINUITY AND IRRATIONAL NUMBERS.

MY attention was first directed toward the considerations which form the subject of this pamphlet in the autumn of 1858. As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic.

. . . . . .

The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given; even the most rigorous expositions of the differential calculus do not base their proofs upon continuity but, with more or less consciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner.

Before presenting the arithmetic basis, Dedekind describes the process to be employed.

Just as negative and fractional rational numbers are formed by a new creation, and as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone. The question only remains how to do this.

Question

5. Suggest a way to define the negative numbers in terms of the natural numbers, or the rational numbers in terms of integers.

With the central object of defining the irrationals, Dedekind first notes properties of the rational numbers.

I. If $a > b$, and $b > c$, then $a > c$.

II. If a, c are two different numbers, there are infinitely many different numbers lying between a, c.

III. If a is any definite number, then all numbers of the system R fall into two classes, A_1 and A_2, each of which contains infinitely many individuals; the first class A_1 comprises all numbers a_1 that are $< a$, the second class A_2 comprises all numbers a_2 that are $> a$; the number a itself may be assigned at pleasure to the first or second class, being respectively the greatest number of the first class or the least of the second. In every case the separation of the system R into the two classes A_1, A_2 is such that every number of the first class A_1 is less than every number of the second class A_2.

Dedekind then draws a parallel between these properties of the rational numbers and properties of points on a line L.

I. If p lies to the right of q, and q to the right of r, then p lies to the right of r; and we say that q lies between the points p and r.

II. If p, r are two different points, then there always exist infinitely many points that lie between p and r.

III. If p is a definite point in L, then all points in L fall into two classes, P_1, P_2, each of which contains infinitely many individuals; the first class P_1 contains all the points p_1, that lie to the left of p, and the second class P_2 contains all the points p_2 that lie to the right of p; the point p itself may be assigned at pleasure to the first or second class. In every case the separation of the straight line L into the two classes or portions P_1, P_2, is of such a character that every point of the first class P_1 lies to the left of every point of the second class P_2.

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* Dedekind denotes the set of rational numbers by R

This analogy between rational numbers and the points of a straight line, as is well known, becomes a real correspondence when we select upon the straight line a definite origin or zero-point o and a definite unit of length for the measurement of segments.

Questions

6. In property III (both for the rationals and the points on L), what sets are represented by

$$A_1 \cup A_2 \quad (\text{or } P_1 \cup P_2)$$

$$A_1 \cap A_2 \quad (\text{or } P_1 \cap P_2) \text{ ?}$$

7. Which of the following could be true? Explain. (In sections ii) to vii) $a_1 \in A_1$, $a_2 \in A_2$.)

   i) $A_1 = \phi$

   ii) $a_2 = a_1$

   iii) $a_1 < a_2$

   iv) $a_1 > a$

   v) $a_2 > a$

   vi) $a_1 = a$

   vii) $a_2 = a$

8. Prove that, if there exists a number $a_1 \in A_1$ which is greater than any other number in A_1, then there does not exists a number $a_2 \in A_2$, which is smaller than every other number in A_2.

Dedekind now continues as follows.

> In the preceding section attention was called to the fact that every point p of the straight line produces a separation of the same into two portions such that every point of one portion lies to the left of every point of the other. I find the essence of continuity in the converse, i. e., in the following principle:
>
> "If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."

Question

9. Rewrite the axiom of continuity, given in terms of a line and the points which constitute it, in terms of *the system* R of rational numbers, and check whether you are willing to accept the latter also as an axiom.

The next stage in the process, as given by Dedekind, is:

CREATION OF IRRATIONAL NUMBERS.

> From the last remarks it is sufficiently obvious how the discontinuous domain R of rational numbers may be rendered complete so as to form a continuous domain.
>
> ... any separation of the system R into two classes A_1, A_2, is given which possesses only *this* characteristic property that every number a_1 in A_1 is less than every number a_2 in A_2, then for brevity we shall call such a separation a *cut* [Schnitt] and designate it by (A_1, A_2). ...
>
> But it is easy to show that there exist infinitely many cuts not produced by rational numbers.

Thus, the Dedekind cuts define the irrational numbers as follows.

- if (A_1, A_2) is such that A_1 has a greatest number a, (or A_2 has a least number a), then the cut is just another label for the rational number a.
- if no such a exists then the cut is a label for an irrational number - that is, the irrationals are now defined as "cuts" of the rationals.

10. Define $\sqrt{2}$.

11. Define π .

Having completed the definition of the irrationals (and with the rationals also relabeled in the same terms as cuts), Dedekind remarks

> In order to obtain a basis for the orderly arrangement of all *real*, i. e., of all rational and irrational numbers we must investigate the relation between any two cuts (A_1, A_2) and (B_1, B_2) ...

12. We write $\alpha = (A_1, A_2)$, $\beta = (B_1, B_2)$.

Define

i) $\alpha = \beta$

ii) $\alpha > \beta$

iii) $\alpha + \beta$

iv) $\alpha \cdot \beta$, and *prove* that $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$.

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The Irrational Numbers

Answer sheet: Dedekind and the definition of the irrationals.

1-4. Dedekind's definition of the irrationals is based on the rationals and some of their properties. Therefore, we begin by summarising some of the important properties of the latter.

Suppose that a is the rational number closest to 1/2 (either greater or less than 1/2). Then

$b = \dfrac{a + \frac{1}{2}}{2}$ is also rational.

But $\left|\frac{1}{2} - b\right| = \left|\frac{1}{2} - \dfrac{a + \frac{1}{2}}{2}\right| = \left|\dfrac{\frac{1}{2} - a}{2}\right| < \left|\frac{1}{2} - a\right|$.

Hence b is nearer to 1/2 than a.

It follows that between any two rationals there is an infinity of rationals. But in spite of their "density", there are further "numbers" which are not rationals. For example, in the interval [0.7, 0.71], the "number" $\frac{\sqrt{2}}{2}$ (0.7071...), is not rational.

Similar arguments lead us to the conclusion that there does not exist a number, rational or not, which is closest to $\sqrt{2}$ (see, for example, the Bombelli worksheet).

5. In the set of worksheets on the history of the negative numbers, we brought one way of defining the negative numbers. The rationals can be defined analogously. See also, Richardson, M. *Fundamentals of Mathematics*, Macmillan, 1960, pp. 78-86.

6-7. The purpose of these questions is to express Dedekind's text in modern symbolism, and thus to make sure that the text is understood.

In property III

$A_1 \cup A_2 = R$ (rationals) (or $P_1 \cup P_2 = L$)

$A_1 \cap A_2 = \emptyset$ (or $P_1 \cap P_2 = \emptyset$)

i) This is false. For if $A_1 = \emptyset$, then $A_2 = R$, and we do not have a separation of R.

ii) False, since $A_1 \cap A_2 = \emptyset$.

iii) True.

iv) False, since every number greater than *a* belongs to A_2.

v) True.

vi)-vii) "the number *a* itself may be assigned at pleasure to the first or second class". Thus *one* of these two statements may be true, but not both.

8. Suppose, on the contrary that A_1 has a greatest number a_1 and that A_2 has a least number a_2. The

$$\frac{a_1 + a_2}{2}$$

is a rational number, such that

$$a_1 < \frac{a_1 + a_2}{2} < a_2$$

and hence belongs to neither A_1 or A_2 - which contradicts the definition of separation.

9. Dedekind states the axiom of continuity as follows:

> **"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."**

Let us rewrite it in terms of *the system R* of rational numbers:

"If all *numbers* of the *system R* fall into two classes such that every *number* of the first class *is less than* every *number* of the second class, then there exists only one *number* which produces this division of all *numbers* into two classes, this severing of *the system R* into two portions".

Since "number" here means "rational number", this statement is not only not an axiom, but it is demonstrably false. The following is an example of a separation of R in the sense of Dedekind, which is not defined by any (rational) number.

A_2 is the set of all positive rationals whose square is greater than 2,

and A_1 the remaining rationals.

10. (A_1, A_2) as given at the end of question 9, define $\sqrt{2}$.

11. In the previous question we used an "intuitive" property of $\sqrt{2}$ (i.e. that its "square" is 2) to help us find the appropriate cut, which then *becomes the definition of* $\sqrt{2}$. Logically, of course, we do not know what $\sqrt{2}$ is until it is defined; therefore, the question, as stated is illegitimate.

The same applies to this question. What is π ? Until we have defined it, we do not know what it is. The intention, however, is clear. The question is asking for the determination of a cut which corresponds to our previous intuitive knowledge of π. This is far from easy: any "formula" for π involves consideration of some infinite process, which in turn requires the mathematical concept of convergence. And it was precisely for this reason, among others, that Dedekind felt the need to supply a pure mathematical definition of continuity in arithmetic; that is, to define the real numbers.

Having developed the concept of convergence, one can prove, for example, that the series

$$4(1 - \frac{1}{3} + \frac{1}{7} - \frac{1}{9} + \ldots .)$$

converges. The sum of this series may then be taken to be the mathematical definition of π.

It is then superfluous to find the Dedekind cut which (also) defines π, but this may be done by comparing any rational with the sum of the series, and assigning it to one of two classes according to whether it is greater or less than this sum.

12. For each of the required definitions, it is sufficient to determine the conditions or properties of one class; the second class is then the remainder of the rational numbers. Also (see remarks in answer to question 11), we may clearly choose definitions at will - unless they lead to contradictions. But this is rarely a worthwhile exercise. The implication in the question is that we should be guided by our previous intuitive experience and, in particular, that the definitions, which will also apply to the rationals since they are included in the cuts, should accord with those definitions already given for the same concepts or operations for the rational numbers.

We use the notation $\alpha = (A_1, A_2)$, $\beta = (B_1, B_2)$

i) The "obvious" definition would seem to be

$$\alpha = \beta \iff A_1 = B_1 ,$$

but, unfortunately, this is "wrong". For, if the cut is rational (i.e. defined by a rational number), it could happen that the rational number, in one case, belongs to the first class, and in the second case to the second class, and nevertheless the cuts must be regarded as the same, since they define the same rational number.

Therefore, we define

$$\alpha = \beta \iff A_1 = B_1 ,$$

or if there is one and only one element in A_1 (B_1) which is not in B_1 (A_1) .

ii) If A_1 contains more than one rational number that are not contained in B_1, then we define

$$\alpha > \beta \iff A_1 \supset B_1 .$$

It can be proved that the relation $>$, as here defined, has all the properties which are already familiar from the set of rationals.

iii) $$\alpha + \beta = \gamma$$

where $\gamma = (C_1 , C_2)$ and is defined by

$$C_1 = \left\{ c_1 \mid a_1 + b_1 = c_1 , a_1 \in A_1 , b_1 \in B_1 \right\}$$

iv) To define $\alpha \cdot \beta = \gamma$, we have to distinguish between a number of cases

- if $\alpha > 0$, $\beta > 0$ (according to the definition in section ii), $\alpha > 0$ if A_1 contains rationals greater than zero)

then we define

$$C_2 = \left\{ c_2 | c_2 = a_2 b_2 , a_2 \in A_2 , b_2 \in B_2 \right\}$$

- if $\alpha < 0$, $\beta < 0$, then

$$C_2 = \left\{ c_2 | c_2 = a_1 b_1 , a_1 \in A_1 , b_1 \in B_1 \right\}$$

- if $\alpha < 0$, $\beta > 0$, then

$$C_1 = \left\{ c_1 | c_1 = a_1 b_2 , a_1 \in A_1 , b_2 \in B_2 \right\}$$

The reason for these separate cases, should be clear. If not consider a numerical example of each.

To prove that $\sqrt{2}\cdot\sqrt{3} = \sqrt{6}$, we first note that

$$\sqrt{2} = (A_1 , A_2), \text{ where } A_2 = \left\{x \mid x > 0 \text{ and } x^2 > 2\right\}$$

$$\sqrt{3} = (B_1 , B_2), \text{ where } B_2 = \left\{y \mid y > 0 \text{ and } y^2 > 3\right\}$$

By the first case in our definition of the product, we have

$$\sqrt{2}\cdot\sqrt{3} = (C_1 , C_2) , \text{ where } C_2 = \left\{z \mid z = xy,\ x\varepsilon A_2,\ y\varepsilon B_2\right\}$$

Since $x > 0$ and $y > 0$, it follows that $z = xy > 0$

Also $x^2 > 2$ and $y^2 > 3$ $\qquad z^2 = (xy)^2 = x^2y^2 > 6$

Hence the cut (C_1 , C_2) corresponds to $\sqrt{6}$.

Note 1 In this worksheet we have seen how Dedekind "completes" the set R (rationals) to obtain the real numbers. Dedekind, himself, continues by proving that in the extended set we have obtained the desired parallelism between numbers and points:

Every real number produces a division of the set into two classes and every division is formed by some real number. (Remember, it is the second half of this statement which fails in the case of the rationals.)

Note 2 There are alternative definitions of the real numbers. For example, Cantor is credited with the definition using infinite sequences of rational numbers. (See, for example, Goffman, C., *Real Functions*, Rinehart, 1953, p. 28-41.)

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